

ON THE EXISTENCE OF POSITIVE SOLUTIONS FOR SOME NONLINEAR BOUNDARY VALUE PROBLEMS II

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ABSTRACT. We study a class of boundary value problems with φ -Laplacian (e.g., the prescribed mean curvature equation, in which $\varphi(s) = \frac{s}{\sqrt{1+s^2}}$)

$$-(\varphi(u'))' = \lambda f(u) \text{ on } (-L, L), \quad u(-L) = u(L) = 0,$$

where λ and L are positive parameters. For convex f with $f(0) = 0$, we establish various results on the exact number of positive solutions as well as global bifurcation diagrams. Some new bifurcation patterns are shown. This paper is a continuation of [13], where the case $f(0) > 0$ has been investigated.

1. INTRODUCTION

Consider the following nonlinear boundary value problem

$$\begin{cases} -(\varphi(u'))' = \lambda f(u), & x \in (-L, L), \\ u(-L) = u(L) = 0, \end{cases} \quad (1.1)$$

where φ and f satisfy the conditions

$$\varphi \in C^2(\mathbb{R}) \text{ is odd and } \varphi'(t) > 0 \text{ for all } t \in \mathbb{R}. \quad (1.2)$$

$$\begin{aligned} f &\text{ is continuous on } [0, A) \text{ satisfying } f(u) > 0 \text{ for all } 0 < u < A, \\ &\text{where either } A = +\infty, \text{ or } A < +\infty \text{ with } \lim_{u \rightarrow A^-} f(u) = +\infty. \end{aligned} \quad (1.3)$$

This paper is a continuation of the paper by Pan and Xing [13], where various results on the existence and exact number of positive solutions are obtained when f is an increasing convex function with $f(0) > 0$. In the present paper, we investigate the case $f(0) = 0$.

For various different purposes, φ and f are also required to satisfy some or all of the conditions:

$$z\varphi''(z) \leq 0 \text{ for all } z \in \mathbb{R}, \quad (1.4)$$

$$f \text{ is of class } C^1 \text{ on } [0, A) \text{ satisfying } f'(u)u \geq f(u) \text{ for } u \in (0, A). \quad (1.5)$$

$$\begin{aligned} &\text{One of the inequalities in (1.4) and (1.5) is strict, except for at most} \\ &\text{a finite number of } z \text{ or } u. \end{aligned} \quad (1.6)$$

The problem (1.1) with (1.2) and (1.4) includes many important examples such as

$$\begin{cases} -\frac{u''}{(1+|u'|^2)^{\frac{k}{2}}} = \lambda f(u), & x \in (-L, L), \\ u(-L) = u(L) = 0, \end{cases} \quad (1.7)$$

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where $k \geq 0$ and $\varphi(s) = \int_0^s (1+t^2)^{-\frac{k}{2}} dt$. When $k = 2$, (1.7) becomes

$$\begin{cases} -\frac{u''}{1+|u'|^2} = \lambda f(u), & x \in (-L, L), \\ u(-L) = u(L) = 0, \end{cases} \quad (1.8)$$

and $\varphi(s) = \arctan s$. Problem (1.8) is related to a MEMS model with fringing field (see e.g. [18]). When $k = 3$, (1.7) becomes one-dimensional prescribed mean curvature problem

$$\begin{cases} -\left(\frac{u'}{\sqrt{1+|u'|^2}}\right)' = \lambda f(u), & x \in (-L, L), \\ u(-L) = u(L) = 0, \end{cases} \quad (1.9)$$

and $\varphi(s) = \frac{s}{\sqrt{1+s^2}}$. Quasilinear problem (1.9) absorbed much attention in recent years and some special nonlinearities f satisfying $f(0) = 0$ such as $u^p, e^u - 1, u^p + u^q$, were studied and many interesting results on existence and exact multiplicity were obtained (see [1, 2, 4, 5, 6, 8, 11, 12, 14, 15, 19]).

In [13], due to limitations of space, we only focused on the case $f(0) > 0$. In the present paper, we still use the time-map method, following the same line as in [13], to investigate the equally important case $f(0) = 0$. This method is based on the fact that the investigation of the exact number of positive solutions of problem (1.1) is equivalent to studying the shape of a time map T . We will further reduce the problem to the shape of a simpler function g , which describes the values of T at the right endpoint of the interval of definition. The patterns of bifurcation diagrams for problem (1.1) with (1.2)–(1.6) finally depend on the number of local extreme points and the local extreme values of g . By analytical proof or numerical simulation, we find many interesting new examples, which suggest that there exist more than ten types of shapes of g . This means that bifurcation diagrams of (1.1) can contain very complex patterns. We establish various results on the exact number of positive solutions as well as bifurcation diagrams corresponding to some different types of g . Although these results and figures occupy much space of the present paper, we think that it is worth doing in order to show important details.

In this paper, by a *positive solution* we mean a positive classical solution, that is, a function $u \in C^2[-L, L]$ satisfying (1.1) and $u > 0$ in $(-L, L)$.

We organize the paper as follows. In Section 2, we present our main results about the existence and exact number of positive solutions as well as global bifurcation diagrams. In Section 3, we investigate general properties of the time map. The proofs of the main results will be given in Section 4.

2. MAIN RESULTS

Our main results are the following theorems. Notice that (1.3) and (1.5) imply that $f(0) = 0$ and both $f(u)$ and $\frac{f(u)}{u}$ are increasing for $u \in (0, A)$.

Theorem 2.1. *Assume conditions (1.2)–(1.6) hold. Then (1.1) has at most one positive solution for any $\lambda > 0$.*

Since $f''(u) \geq (>)0$ and $f(0) = 0$ imply $f'(u)u \geq (>)f(u)$, we have

Corollary 2.2. *Replacing (1.5) and (1.6) in Theorem 2.1 by the following conditions*

$$f \text{ is of class } C^1 \text{ on } [0, A) \text{ and } C^2 \text{ on } (0, A) \text{ satisfying } f''(u) \geq 0 \text{ and } f(0) = 0. \quad (1.5')$$

$$\text{One of the inequalities in (1.4) and (1.5')} \text{ is strict, except for at most a finite number of } z \text{ or } u. \quad (1.6')$$

Then the conclusion of Theorem 2.1 is still true.

Remark 2.1. (a) Conditions (1.5) and (1.5') in the above results are crucial. For example, for problem (1.9) with $f(u) = u^p$ ($0 < p < 1$) or $f(u) = u - u^3$, it is well known in [6] or [5] that there exists $\lambda^* > 0$ such that (1.1) has exactly two positive solutions.

(b) Condition (1.6) or (1.6') cannot be removed from the above results. For example, the linear eigenvalue problem

$$-u'' = \lambda u \text{ on } (-L, L), \quad u(-L) = u(L) = 0$$

has infinitely many positive solutions for $\lambda = (\frac{\pi}{2L})^2$. Moreover, if f satisfies (1.5) and there exists an r_0 such that $f(w) = m_0 w$ for $0 \leq w \leq r_0$, then the problem

$$-u'' = \lambda f(u) \text{ on } (-L, L), \quad u(-L) = u(L) = 0$$

also has infinitely many positive solutions for $\lambda = \frac{1}{m_0} (\frac{\pi}{2L})^2$ (see e.g. [9, Thm 3.2]).

In order to further give the exact number of positive solutions for each λ , we introduce some notations. The same as in [13], we denote

$$\Phi(z) = \int_0^z t \varphi'(t) dt \quad \text{and} \quad F(u) = \int_0^u f(s) ds.$$

and define

$$B = \sup_{z \in [0, +\infty)} \Phi(z) \quad \text{and} \quad C = \sup_{u \in [0, A)} F(u).$$

Then (1.2) and (1.3) imply that $B = \lim_{z \rightarrow +\infty} \Phi(z)$ and $C = \lim_{u \rightarrow A^-} F(u)$, respectively.

Since (1.3) and (1.5) implies $f' \geq 0$, it follows that $A = +\infty$ implies $C = +\infty$. The same as in [13], the problem considered in Theorem 2.1 can be classified into the following six cases.

$$\left. \begin{array}{l} \text{Six cases of (1.1)} \end{array} \right\} \begin{array}{l} B = +\infty \left\{ \begin{array}{ll} A = +\infty & C = +\infty \quad (\text{Case I}) \\ A < +\infty & \left\{ \begin{array}{ll} C = +\infty & (\text{Case II}) \\ C < +\infty & (\text{Case III}) \end{array} \right. \end{array} \right. \\ \\ B < +\infty \left\{ \begin{array}{ll} A = +\infty & C = +\infty \quad (\text{Case IV}) \\ A < +\infty & \left\{ \begin{array}{ll} C = +\infty & (\text{Case V}) \\ C < +\infty & (\text{Case VI}) \end{array} \right. \end{array} \right. \end{array} \quad (*)$$

We shall mainly focus on the situation where the range of φ is bounded. However, Theorems 2.3–2.5 and many results in Sections 3 and 4 are also of interest when φ is unbounded. Notice that under (1.2), condition $B < +\infty$ implies that φ is bounded ([13, Remark 2.2]).

Example 2.2 ([13]). Denote $\varphi_k(s) = \int_0^s (1+t^2)^{-\frac{k}{2}} dt$ ($k \geq 0$). Then φ_k satisfies both (1.2) and (1.4). Moreover, we have

φ	$B = +\infty$	$B < +\infty$
Bounded	$\varphi_2(s) = \arctan s$ (Problem (1.8))	$\varphi_k(s)$, $k > 2$, e.g. $\varphi_3(s) = \frac{s}{\sqrt{1+s^2}}$ (Mean Curvature Type) $\varphi_5(s) = \frac{s}{\sqrt{1+s^2}} - \frac{1}{3} \frac{s^3}{(1+s^2)^{\frac{3}{2}}}$
Unbounded	$\varphi_k(s)$, $0 \leq k < 2$	

When $k > 2$, we also have

$$\Phi_k(z) = \frac{1}{k-2} - \frac{1}{k-2} \frac{1}{(1+z^2)^{\frac{k-2}{2}}}, \quad B = \frac{1}{k-2}, \quad \Phi_k^{-1}(y) = \frac{\sqrt{1 - [1 - (k-2)y]^{\frac{2}{k-2}}}}{[1 - (k-2)y]^{\frac{1}{k-2}}}.$$

We next investigate Cases I–VI in (*), respectively. Denote $\lambda_1 = \frac{\varphi'(0)}{f'(0)} (\frac{\pi}{2L})^2$. First, we consider the three cases of $B = +\infty$.

Case I: $B = +\infty$, $A = +\infty$ and $C = +\infty$

Theorem 2.3 (Type I, see Fig.1). *Let $A, B, C = +\infty$. Assume conditions (1.2)–(1.6) hold. Also assume*

$$\lim_{t \rightarrow +\infty} \frac{\varphi \circ \Phi^{-1}(\lambda t)}{f \circ F^{-1}(t)} = 0 \quad \text{for any } \lambda > 0. \quad (2.1)$$

Then the following assertions hold:

- (a) If $f'(0) = 0$, then (1.1) has exactly one positive solution for any $\lambda \in (0, +\infty)$.
- (b) If $f'(0) > 0$, then (1.1) has exactly one positive solution for $\lambda \in (0, \lambda_1)$ and none for $\lambda \in [\lambda_1, +\infty)$.

Corollary 2.4. *Let $A, B, C = +\infty$. Assume conditions (1.2)–(1.6) hold. If the range of φ is bounded or $\frac{F(z)}{f(z)}$ is bounded for sufficiently large z , then (2.1) holds and hence the conclusions of Theorem 2.3 hold.*

Example 2.3. Let $\varphi = \varphi_k$ ($0 \leq k \leq 2$), which is defined in Example 2.2, and let f be one of the following table.

f	$\frac{F(z)}{f(z)}$ is unbounded	$\frac{F(z)}{f(z)}$ is bounded
f convex, $f(0) = 0$, $A, C = +\infty$	$f(u) = u^p, p \geq 1$ $f(u) = u^p + u^q, q > p \geq 1$ $f(u) = (1+u)^p - 1, p > 1$	$f(u) = e^u - 1$ $f(u) = e^u + u^p - 1, p \geq 1$ $f(u) = e^{u^2} - 1$ $f(u) = e^{u^2} + u^p - 1, p \geq 1$

The following two groups of φ and f give some examples which satisfy the conditions of Corollary 2.4:

- (1) φ_2 and any f in the above table;
- (2) $\varphi = \varphi_k$ ($0 \leq k < 2$) and any f in the last column of the above table.

Case II: $B = +\infty$, $A < +\infty$ and $C = +\infty$

Theorem 2.5 (see Fig.1). *Let $A < +\infty$, $B = +\infty$, $C = +\infty$. Assume conditions (1.2)–(1.6) hold. Then the following assertions hold:*

- (a) If $f'(0) = 0$, then (1.1) has exactly one positive solution for any $\lambda \in (0, +\infty)$.
- (b) If $f'(0) > 0$, then (1.1) has exactly one positive solution for $\lambda \in (0, \lambda_1)$ and none for $\lambda \in [\lambda_1, +\infty)$.

Case III: $B = +\infty$, $A < +\infty$ and $C < +\infty$

Theorem 2.6 (see Fig.1). *Let $A < +\infty$, $B = +\infty$, $C < +\infty$. Assume conditions (1.2)–(1.6) hold. Then the following assertions hold:*

- (a) If $f'(0) = 0$, then there exists $\lambda_* > 0$ such that (1.1) has exactly one positive solution for $\lambda \in (\lambda_*, +\infty)$ and none for $(0, \lambda_*]$.
- (b) If $f'(0) > 0$, then there exists $\lambda_* \in (0, \lambda_1)$ such that (1.1) has exactly one positive solution for $\lambda \in (\lambda_*, \lambda_1)$ and none for $\lambda \in (0, \lambda_*] \cup [\lambda_1, +\infty)$.
- (c) λ_* is strictly decreasing with respect to L .

Example 2.4. Let $\varphi = \varphi_k$ ($0 \leq k \leq 2$), which is defined in Example 2.2, and let f be one of the following table.

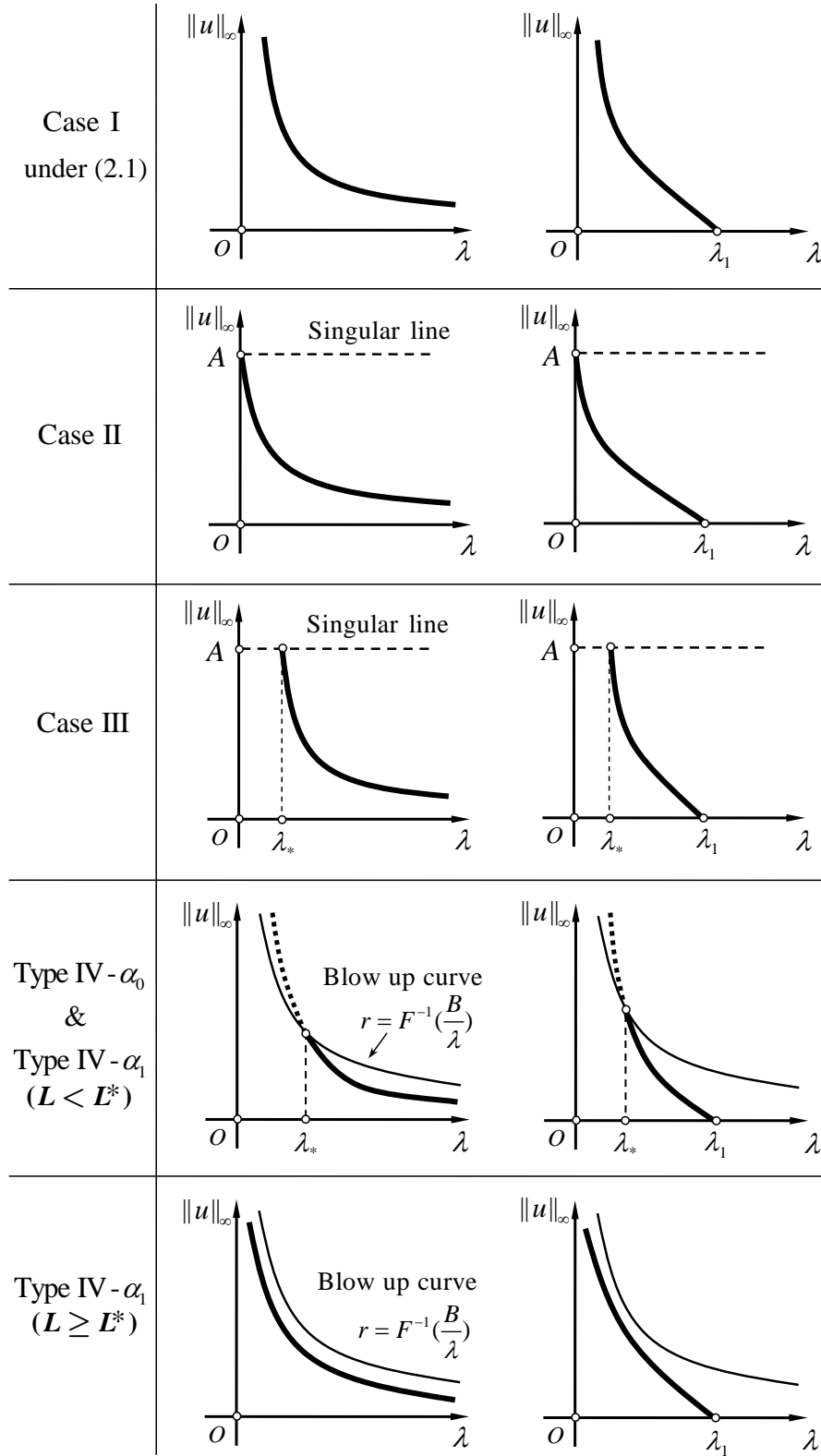


FIGURE 1. Bifurcation Diagrams for Cases I-III, Types IV- α_0 and IV- α_1 with $f(0) = 0$. Left: $f'(0) = 0$. Right: $f'(0) > 0$.

f	$f(0) = 0, f'(0) = 0$	$f(0) = 0, f'(0) > 0$
$A < +\infty, C = +\infty$ (Case II)	$f(u) = \tan u^q, u \in [0, \sqrt[q]{\frac{\pi}{2}}) (q > 1)$ $f(u) = (1 - u^q)^{-p} - 1, u \in [0, 1)$ ($q > 1, p \geq 1$)	$f(u) = \tan u, u \in [0, \frac{\pi}{2})$ $f(u) = (1 - u)^{-p} - 1, u \in [0, 1)$ ($p \geq 1$)
$A, C < +\infty$ (Case III)	$f(u) = (1 - u^q)^{-p} - 1, u \in [0, 1)$ ($q > 1, 0 < p < 1$)	$f(u) = (1 - u)^{-p} - 1, u \in [0, 1)$ ($0 < p < 1$)

Then φ and f give various examples which satisfy the conditions of Theorem 2.5 or 2.6.

We next consider the three cases of $B < +\infty$. The same as in [13], we introduce the function

$$g(\lambda) = \int_0^{F^{-1}(\frac{B}{\lambda})} \frac{1}{\Phi^{-1}(B - \lambda F(u))} du. \quad (2.2)$$

As pointed out in [13], for given φ and f , the function g plays a crucial role in determining bifurcation diagrams of problem (1.1): when the length parameter L passes through the local extreme values of g , the pattern of the bifurcation diagram must change; the complexity of the graph of g leads to the rich diversity of bifurcation patterns for (1.1).

To distinguish different types of g , we introduce the following definitions.

Definition 2.5 (See Fig.2). We say that the function g is of

Type γ_2 , if $\lim_{\lambda \rightarrow +\infty} g(\lambda) = 0, \lim_{\lambda \rightarrow 0} g(\lambda) \in (0, +\infty)$, $g(\lambda)$ has exactly two local extreme points in $(0, +\infty)$

and the local maximum value is greater than $\lim_{\lambda \rightarrow 0} g(\lambda)$;

Type γ_3 , if $\lim_{\lambda \rightarrow +\infty} g(\lambda) = 0, \lim_{\lambda \rightarrow 0} g(\lambda) \in (0, +\infty)$, $g(\lambda)$ has exactly two local extreme points in $(0, +\infty)$

and the local maximum value is equal to $\lim_{\lambda \rightarrow 0} g(\lambda)$;

Type δ_2 , if $\lim_{\lambda \rightarrow +\infty} g(\lambda) = 0, \lim_{\lambda \rightarrow 0} g(\lambda) = 0$, $g(\lambda)$ has exactly three local extreme points in $(0, +\infty)$

and the left local maximum value is greater than the right one;

Type δ_3 , if $\lim_{\lambda \rightarrow +\infty} g(\lambda) = 0, \lim_{\lambda \rightarrow 0} g(\lambda) = 0$, $g(\lambda)$ has exactly three local extreme points in $(0, +\infty)$

and the left local maximum value is equal to the right one;

\dots .

Among the graphs in Fig.2, the remaining, unmentioned types in Definition 2.5 have been introduced in [13]. Here, we use $\alpha, \beta, \gamma, \delta, \dots$ (in the Greek alphabetical order) to represent the numbers of the local extremum points of g in $(0, +\infty)$ being zero, one, two, three, \dots , respectively.

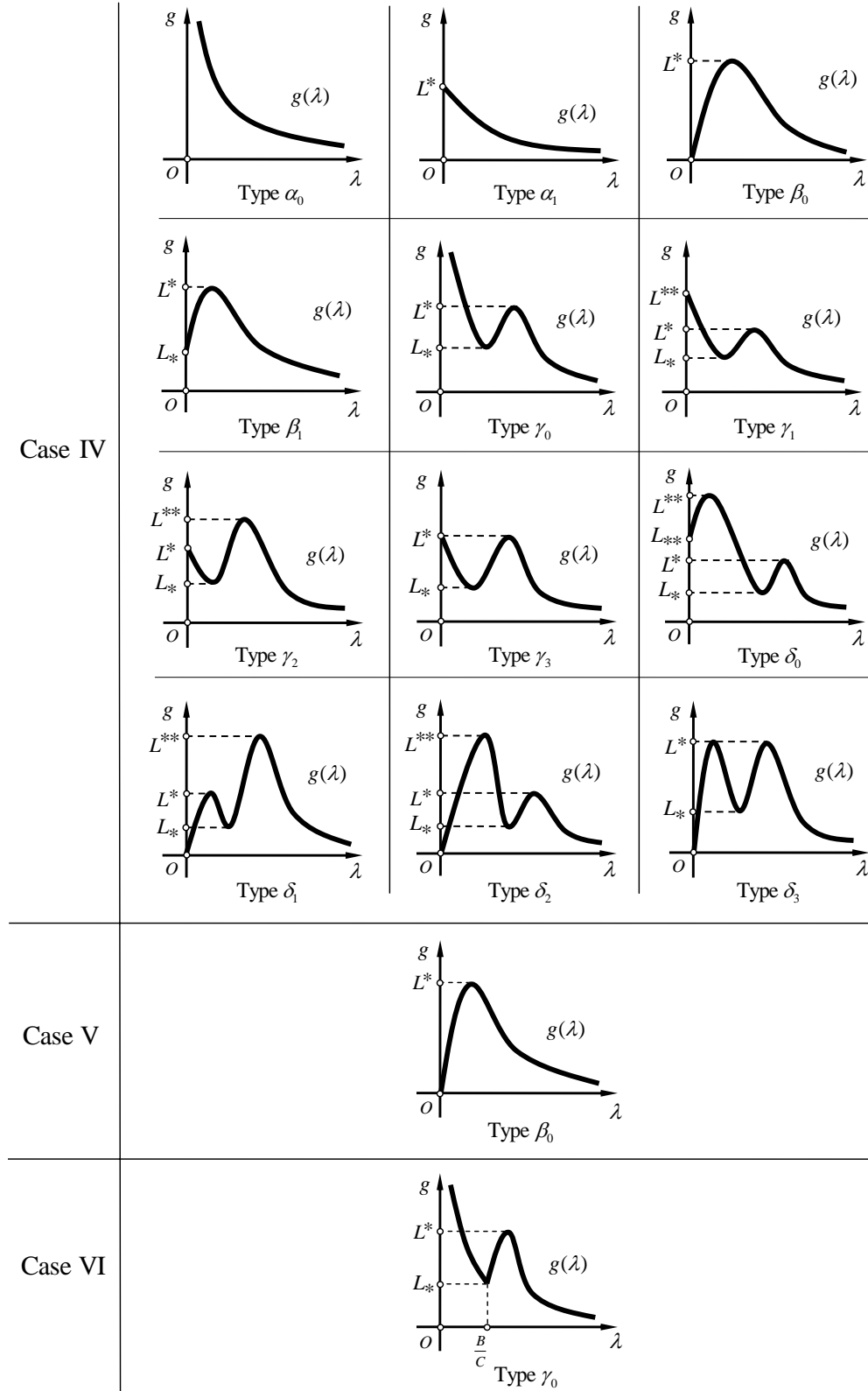
Definition 2.6. If φ and f in (1.1) satisfy conditions $A = +\infty, B < +\infty$ and $C = +\infty$, i.e., Case IV, and g is of Type γ_2 ($\gamma_3, \delta_2, \dots$, respectively), then we say the pair (φ, f) is of Type IV- γ_2 (IV- γ_3 , IV- δ_2, \dots , respectively).

In order to determine the type of g , the same as in [13], we introduce the following conditions

$$K := \int_0^B \frac{1}{y\Phi^{-1}(B - y)} dy < +\infty. \quad (2.3)$$

$$f'(z)F(z) \lesssim f^2(z) \quad \text{for } z \in (0, A). \quad (2.4)$$

Here, the notation " \lesssim ", which lies in between " \leq " and " $<$ ", means that except for at most finitely many points where " $=$ " holds, it is always " $<$ ".

FIGURE 2. Some shapes of the function g for $B < +\infty$.

Example 2.7 ([13]). Consider the function $\varphi_k(s) = \int_0^s (1+t^2)^{-\frac{k}{2}} dt$ which is introduced in Example 2.2. A direct computation shows that (2.3) is satisfied for all $k > 2$, e.g., $K = \frac{\pi}{2}$ when $k = 3$ (i.e., the mean curvature equation).

Example 2.8. The following table gives some functions f satisfying both $f(0) = 0$ and condition (2.4).

f	$f(0) = 0$
$f'(u)F(u) < f^2(u)$	$f(u) = (1+u)^p - 1, p > 0$
	$f(u) = e^u - 1$
	$f(u) = u^p, p > 0$
	$f(u) = u^p + u^q, 0 \leq p < q < \tilde{q}(p)$
$f'(u)F(u) \lesssim f^2(u)$	$f(u) = u^p + u^q, 0 \leq p < q = \tilde{q}(p)$

Here, the optimal upper bound $\tilde{q}(p) = p + 1 + 2\sqrt{p+1}$ is obtained in [7].

Example 2.9. The following table gives some functions f satisfying both $f(0) = 0$ and $\lim_{t \rightarrow A} \frac{F(t)}{f(t)} = D$.

f	$D = +\infty$	$D \in (0, +\infty)$	$D = 0$
$f(0) = 0$	$f(u) = u^p, p > 0$	$f(u) = e^u - 1$	$f(u) = e^{u^2} - 1$
	$f(u) = u^p + u^q, q > p > 0$	$f(u) = e^u + u^p - 1, p > 0$	$f(u) = e^{u^2} + u^p - 1, p > 0$
	$f(u) = (1+u)^p - 1, p > 0$	$f(u) = u^p e^{ku} + u^q, p, q \geq 1, k > 0$	$f(u) = (1-u)^{-p} - 1, p > 0$

Besides, all of them satisfy $\lim_{t \rightarrow 0} \frac{F(t)}{f(t)} = 0$. These two limits of $\frac{F(t)}{f(t)}$ are very useful for computing the limits of g at 0 and $+\infty$ (see Lemma 3.9 below).

Case IV: $B < +\infty, A = +\infty$ and $C = +\infty$

Theorem 2.7 (Type IV- α_0 , see Fig.1). Let (φ, f) be of Type IV- α_0 (see Fig.2). Assume conditions (1.2)–(1.6) hold. Then the following assertions hold:

- (a) If $f'(0) = 0$, then there exists $\lambda_* > 0$ such that (1.1) has exactly one positive solution for $\lambda \in (\lambda_*, +\infty)$ and none for $\lambda \in (0, \lambda_*]$.
- (b) If $f'(0) > 0$, then there exists $\lambda_* \in (0, \lambda_1)$ such that (1.1) has exactly one positive solution for $\lambda \in (\lambda_*, \lambda_1)$ and none for $\lambda \in (0, \lambda_*] \cup [\lambda_1, +\infty)$.
- (c) λ_* is strictly decreasing with respect to L .

Corollary 2.8 (Type IV- α_0 , see Fig.1). Let $A = +\infty, B < +\infty, C = +\infty$. Assume conditions (1.2)–(1.6) hold. If further (2.3), (2.4) and $\lim_{z \rightarrow A} \frac{F(z)}{f(z)} = +\infty$ hold, then (φ, f) is of Type IV- α_0 and hence the conclusions of Theorem 2.7 hold.

Example 2.10. Let $\varphi = \varphi_k$ ($k > 2$) and $f(u) = u^p$ ($p \geq 1$) or $f(u) = (1+u)^p - 1$ ($p > 1$) or $f(u) = u^p + u^q$ ($1 \leq p < q \leq \tilde{q}(p)$). Then all conditions of Corollary 2.8 are satisfied. Here $\tilde{q}(p)$ is given in Example 2.8. We notice that for $\varphi = \varphi_3$ and $f(u) = u^p + u^q$, this result is obtained in a forthcoming paper [7].

Let us give an explanation of the bifurcation diagrams for Type IV- α_0 in Fig.1. The continuous thick line is the bifurcation curve and represents classical solutions (i.e., $u \in C^2[-L, L]$), while the thin curve $r = F^{-1}(\frac{B}{\lambda})$ is the gradient blow-up curve (see Section 3 for details). We also note that intersection points of bifurcation curves and gradient blow-up curves actually represent another kind of positive solutions which belong to $C[-L, L] \cap C^2(-L, L)$ and satisfy (1.1), but $u'(\pm L) = \mp\infty$. For the other bifurcation diagrams in what follows, we shall use the same legends as explained here (also see Remark 2.25 below).

Theorem 2.9 (Type IV- α_1 , $f'(0) = 0$, see Fig.1). Let (φ, f) be of Type IV- α_1 . Assume conditions (1.2)–(1.6) and $f'(0) = 0$ hold. Then there exists a constant L^* (see Fig.2) such that the following assertions hold:

- (a) If $L < L^*$, there exists $\lambda_* > 0$ such that (1.1) has exactly one positive solution for $\lambda \in (\lambda_*, +\infty)$ and none for $\lambda \in (0, \lambda_*]$. Moreover, λ_* is strictly decreasing with respect to L .
- (b) If $L \geq L^*$, then (1.1) has exactly one positive solution for any $\lambda \in (0, +\infty)$.

Theorem 2.10 (Type IV- α_1 , $f'(0) > 0$, see Fig.1). Let (φ, f) be of Type IV- α_1 . Assume conditions (1.2)–(1.6) and $f'(0) > 0$ hold. Then there exists a constant L^* (see Fig.2) such that the following assertions hold:

- (a) If $L < L^*$, then there exists $\lambda_* \in (0, \lambda_1)$ such that (1.1) has exactly one positive solution for $\lambda \in (\lambda_*, \lambda_1)$ and none for $\lambda \in (0, \lambda_*] \cup [\lambda_1, +\infty)$. Moreover, λ_* is strictly decreasing with respect to L .
- (b) If $L \geq L^*$, then (1.1) has exactly one positive solution for $\lambda \in (0, \lambda_1)$ and none for $\lambda \in [\lambda_1, +\infty)$.

Corollary 2.11 (Type IV- α_1). Let $A = +\infty$, $B < +\infty$, $C = +\infty$. Assume condition (1.2)–(1.6) hold. If further (2.3), (2.4) and $\lim_{z \rightarrow A} \frac{F(z)}{f(z)} \in (0, +\infty)$ hold, then (φ, f) is of Type IV- α_1 and hence the conclusions of Theorem 2.9 or 2.10 hold.

Example 2.11. Let $\varphi = \varphi_k$ ($k > 2$) and $f(u) = e^u - 1$ or $f(u) = e^u - u - 1$. Then all conditions of Corollary 2.11 are satisfied.

Theorem 2.12 (Type IV- β_0 , $f'(0) = 0$, see Fig.3). Let (φ, f) be of Type IV- β_0 . Assume conditions (1.2)–(1.6) and $f'(0) = 0$ hold. Then there exists a constant $L^* > 0$ (see Fig.2) such that the following assertions hold:

- (a) If $L > L^*$, then (1.1) has exactly one positive solution for any $\lambda \in (0, +\infty)$.
- (b) If $L = L^*$, then there exists $\lambda^* > 0$ such that (1.1) has exactly one positive solution for $(0, \lambda^*) \cup (\lambda^*, +\infty)$ and none for $\lambda = \lambda^*$.
- (c) If $L < L^*$, then there exist two numbers $\lambda^* > \lambda_* > 0$ such that (1.1) has exactly one positive solution for $\lambda \in (0, \lambda_*) \cup (\lambda^*, +\infty)$ and none for $\lambda \in [\lambda_*, \lambda^*]$.
- (d) λ^* is strictly decreasing while λ_* is strictly increasing with respect to L .

Theorem 2.13 (Type IV- β_0 , $f'(0) > 0$, see Fig.4). Let (φ, f) be of Type IV- β_0 . Assume conditions (1.2)–(1.6) and $f'(0) > 0$ hold. Then there exists a constant $L^* > 0$ (see Fig.2) such that the following assertions hold:

- (a) If $L > L^*$, then (1.1) has exactly one positive solution for $\lambda \in (0, \lambda_1)$ and none for $\lambda \in [\lambda_1, +\infty)$.
- (b) If $L = L^*$, then there exists $\lambda^* \in (0, \lambda_1)$ such that (1.1) has exactly one positive solution for $(0, \lambda^*) \cup (\lambda^*, \lambda_1)$ and none for $\lambda \in \{\lambda^*\} \cup [\lambda_1, +\infty)$.
- (c) If $L < L^*$, then there exist two numbers $0 < \lambda_* < \lambda^* < \lambda_1$ such that (1.1) has exactly one positive solution for $\lambda \in (0, \lambda_*) \cup (\lambda^*, \lambda_1)$ and none for $\lambda \in [\lambda_*, \lambda^*] \cup [\lambda_1, +\infty)$.
- (d) λ^* is strictly decreasing while λ_* is strictly increasing with respect to L .

Example 2.12. Let $\varphi = \varphi_3$ and $f(u) = e^{u^2} - 1$ or $f(u) = e^{u^2} + u - 1$. Then $\lim_{\lambda \rightarrow +\infty} g(\lambda) = 0 = \lim_{\lambda \rightarrow 0} g(\lambda)$ (by Lemma 3.9 below). Numerical simulation indicates that $g(\lambda)$ has exactly one local extreme point in $(0, +\infty)$ (see Remark 3.1 and Fig.15 below). We provide an analytic proof for this in [17]. Thus g is of Type β_0 and hence both $(\varphi_3, e^{u^2} - 1)$ and $(\varphi_3, e^{u^2} + u - 1)$ are of Type IV- β_0 . From Theorems 2.12 and 2.13, we obtain the exact numbers of positive solutions as well as global bifurcation diagrams.

Theorem 2.14 (Type IV- β_1 , $f'(0) = 0$, see Fig.3). Let (φ, f) be of Type IV- β_1 . Assume conditions (1.2)–(1.6) and $f'(0) = 0$ hold. Then there exist constants $L^* > L_* > 0$ (see Fig.2) such that the following assertions hold:

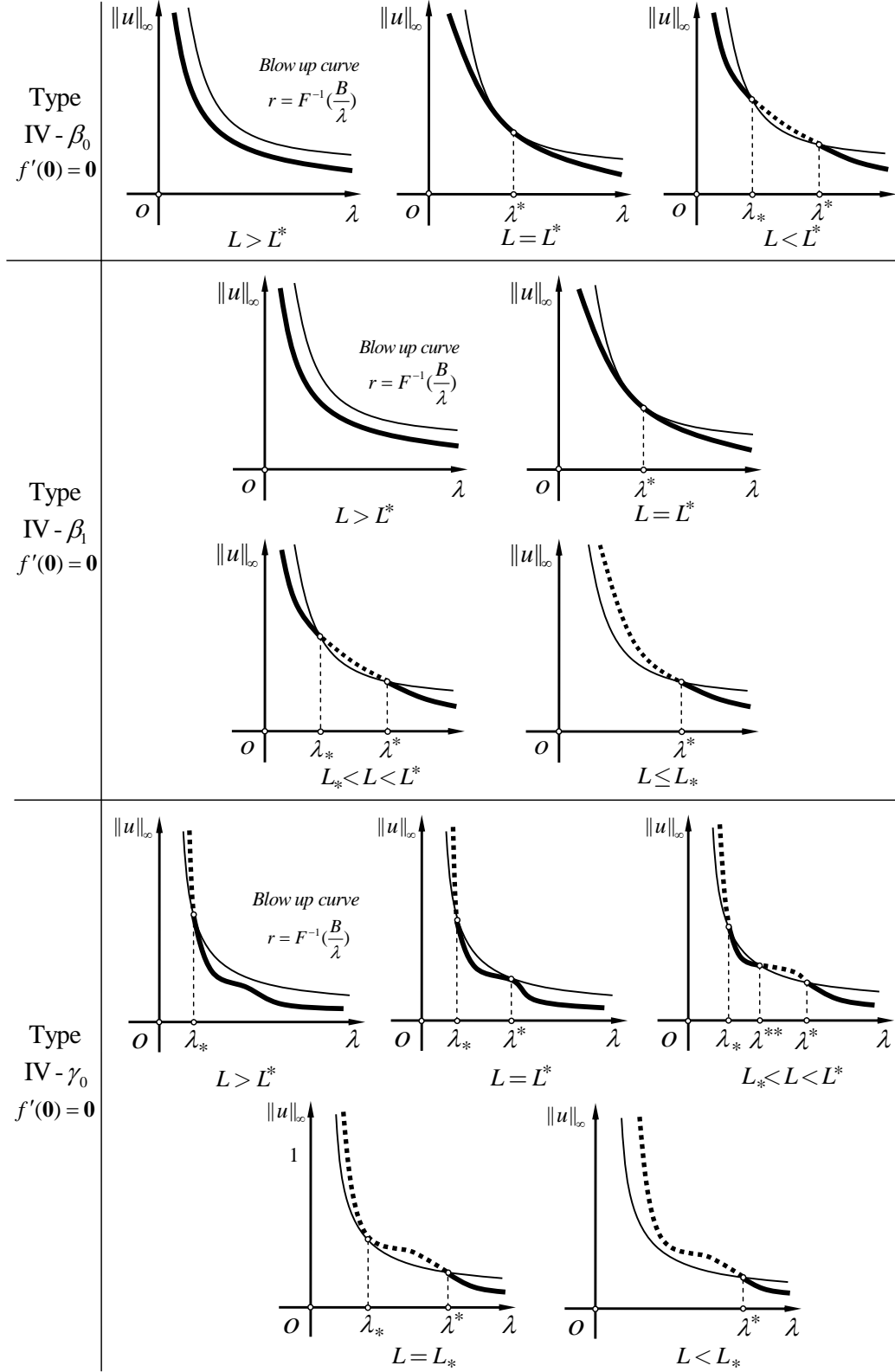


FIGURE 3. Bifurcation Diagrams for Types IV- β_0 , IV- β_1 and IV- γ_0 with $f(0) = 0$ and $f'(0) = 0$.



- (a) If $L > L^*$, then (1.1) has exactly one positive solution for any $\lambda \in (0, +\infty)$.
- (b) If $L = L^*$, then there exists $\lambda^* > 0$ such that (1.1) has exactly one positive solution for $(0, \lambda^*) \cup (\lambda^*, +\infty)$ and none for $\lambda = \lambda^*$.
- (c) If $L_* < L < L^*$, then there exist two numbers $\lambda^* > \lambda_* > 0$ such that (1.1) has exactly one positive solution for $\lambda \in (0, \lambda_*) \cup (\lambda^*, +\infty)$ and none for $\lambda \in [\lambda_*, \lambda^*]$.
- (d) If $L \leq L_*$, then there exists $\lambda^* > 0$ such that (1.1) has exactly one positive solution for $\lambda \in (\lambda^*, +\infty)$ and none for $\lambda \in (0, \lambda^*]$.
- (e) λ^* is strictly decreasing while λ_* is strictly increasing with respect to L .

Theorem 2.15 (Type IV- β_1 , $f'(0) > 0$, see Fig.4). Let (φ, f) be of Type IV- β_1 . Assume conditions (1.2)–(1.6) and $f'(0) > 0$ hold. Then there exist constants $L^* > L_* > 0$ (see Fig.2) such that the following assertions hold:

- (a) If $L > L^*$, then (1.1) has exactly one positive solution for $\lambda \in (0, \lambda_1)$ and none for $\lambda \in [\lambda_1, +\infty)$.
- (b) If $L = L^*$, then there exists $\lambda^* \in (0, \lambda_1)$ such that (1.1) has exactly one positive solution for $(0, \lambda^*) \cup (\lambda^*, \lambda_1)$ and none for $\lambda \in \{\lambda^*\} \cup [\lambda_1, +\infty)$.
- (c) If $L_* < L < L^*$, then there exist two numbers $0 < \lambda_* < \lambda^* < \lambda_1$ such that (1.1) has exactly one positive solution for $\lambda \in (0, \lambda_*) \cup (\lambda^*, \lambda_1)$ and none for $\lambda \in [\lambda_*, \lambda^*] \cup [\lambda_1, +\infty)$.
- (d) If $L \leq L_*$, then there exists $\lambda^* \in (0, \lambda_1)$ such that (1.1) has exactly one positive solutions for $\lambda \in (\lambda^*, \lambda_1)$ and none for $\lambda \in (0, \lambda^*] \cup [\lambda_1, +\infty)$.
- (e) λ^* is strictly decreasing while λ_* is strictly increasing with respect to L .

Example 2.13. Let $\varphi = \varphi_3$ and $f(u) = e^u + u^2 - u - 1$ or $f(u) = e^u + u - 1$. Then $\lim_{\lambda \rightarrow +\infty} g(\lambda) = 0$ and $\lim_{\lambda \rightarrow 0} g(\lambda) = \frac{\pi}{2}$ (by Lemma 3.9 below). Numerical simulation indicates that $g(\lambda)$ has exactly one local extreme point in $(0, +\infty)$ (see Remark 3.1 and Fig.15 below). We provide an analytic proof for this in [17]. Thus g is of Type β_1 and hence both $(\varphi_3, e^u + u^2 - u - 1)$ and $(\varphi_3, e^u + u - 1)$ are of Type IV- β_1 . From Theorems 2.14 and 2.15, we obtain the exact numbers of positive solutions as well as global bifurcation diagrams.

Theorem 2.16 (Type IV- γ_0 , $f'(0) = 0$, see Fig.3). Let (φ, f) be of Type IV- γ_0 . Assume conditions (1.2)–(1.6) and $f'(0) = 0$ hold. Then there exist constants $L^* > L_* > 0$ (see Fig.2) such that the following assertions hold:

- (a) If $L > L^*$, then there exists $\lambda_* > 0$ such that (1.1) has exactly one positive solution for $\lambda \in (\lambda_*, +\infty)$ and none for $\lambda \in (0, \lambda_*]$.
- (b) If $L = L^*$, then there exist two numbers $\lambda^* > \lambda_* > 0$ such that (1.1) has exactly one positive solution for $\lambda \in (\lambda_*, \lambda^*) \cup (\lambda^*, +\infty)$ and none for $\lambda \in (0, \lambda_*] \cup \{\lambda^*\}$.
- (c) If $L_* < L < L^*$, then there exist three numbers $\lambda^* > \lambda^{**} > \lambda_* > 0$ such that (1.1) has exactly one positive solution for $\lambda \in (\lambda_*, \lambda^{**}) \cup (\lambda^*, +\infty)$ and none for $\lambda \in (0, \lambda_*] \cup [\lambda^{**}, \lambda^*]$.
- (d) If $L \leq L_*$, then there exists $\lambda^* > 0$ such that (1.1) has exactly one positive solution for $\lambda \in (\lambda^*, +\infty)$ and none for $\lambda \in (0, \lambda^*]$.
- (e) λ_* and λ^* are strictly decreasing while λ^{**} is strictly increasing with respect to L .

Theorem 2.17 (Type IV- γ_0 , $f'(0) > 0$, see Fig.4). Let (φ, f) be of Type IV- γ_0 . Assume conditions (1.2)–(1.6) and $f'(0) > 0$ hold. Then there exist constants $L^* > L_* > 0$ (see Fig.2) such that the following assertions hold:

- (a) If $L > L^*$, then there exists $\lambda_* \in (0, \lambda_1)$ such that (1.1) has exactly one positive solution for (λ_*, λ_1) and none for $\lambda \in (0, \lambda_*] \cup [\lambda_1, +\infty)$.
- (b) If $L = L^*$, then there exist two numbers $0 < \lambda_* < \lambda^* < \lambda_1$ such that (1.1) has exactly one positive solution for $\lambda \in (\lambda_*, \lambda^*) \cup (\lambda^*, \lambda_1)$ and none for $\lambda \in (0, \lambda_*] \cup \{\lambda^*\} \cup [\lambda_1, +\infty)$.
- (c) If $L_* < L < L^*$, three numbers $0 < \lambda_* < \lambda^{**} < \lambda^* < \lambda_1$ such that (1.1) has exactly one positive solution for $\lambda \in (\lambda_*, \lambda^{**}) \cup (\lambda^*, \lambda_1)$, none for $\lambda \in (0, \lambda_*] \cup [\lambda^{**}, \lambda^*] \cup [\lambda_1, +\infty)$.

(d) If $L \leq L_*$, then there exists $\lambda^* \in (0, \lambda_1)$ such that (1.1) has exactly one positive solution for (λ^*, λ_1) and none for $\lambda \in (0, \lambda^*) \cup [\lambda_1, +\infty)$.

(e) λ_* and λ^* are strictly decreasing while λ^{**} is strictly increasing with respect to L .

Example 2.14. Let $\varphi = \varphi_3$ and $f(u) = u^2 + u^7$ or $f(u) = u + u^6$. Then $\lim_{\lambda \rightarrow +\infty} g(\lambda) = 0$ and $\lim_{\lambda \rightarrow 0} g(\lambda) = +\infty$ (by Lemma 3.9 below). Numerical simulation indicates that $g(\lambda)$ has exactly two local extreme points in $(0, +\infty)$ (see Remark 3.1 and Fig.15 below). We provide an analytic proof for this in [17]. Thus g is of Type γ_0 and hence both $(\varphi_3, u^2 + u^7)$ and $(\varphi_3, u + u^6)$ are of Type IV- γ_0 . From Theorems 2.16 and 2.17, we obtain the exact numbers of positive solutions as well as global bifurcation diagrams.

Theorem 2.18 (Type IV- γ_1 , $f'(0) = 0$, see Fig.5). Let (φ, f) be of Type IV- γ_1 . Assume conditions (1.2)–(1.6) and $f'(0) > 0$ hold. Then there exist constants $L^{**} > L^* > L_* > 0$ (see Fig.2) such that the following assertions hold:

(a) If $L \geq L^{**}$, then (1.1) has exactly one positive solution for any $\lambda \in (0, +\infty)$.

(b) If $L^* < L < L^{**}$, then there exists $\lambda_* > 0$ such that (1.1) has exactly one positive solution for $\lambda \in (\lambda_*, +\infty)$ and none for $\lambda \in (0, \lambda_*]$.

(c) If $L = L^*$, then there exist two numbers $\lambda^* > \lambda_* > 0$ such that (1.1) has exactly one positive solution for $\lambda \in (\lambda_*, \lambda^*) \cup (\lambda^*, +\infty)$ and none for $\lambda \in (0, \lambda_*] \cup \{\lambda^*\}$.

(d) If $L_* < L < L^*$, then there exist three numbers $\lambda^* > \lambda^{**} > \lambda_* > 0$ such that (1.1) has exactly one positive solution for $\lambda \in (\lambda_*, \lambda^{**}) \cup (\lambda^*, +\infty)$ and none for $\lambda \in (0, \lambda_*] \cup [\lambda^{**}, \lambda^*]$.

(e) If $L \leq L_*$, then there exists $\lambda^* > 0$ such that (1.1) has exactly one positive solution for $\lambda \in (\lambda^*, +\infty)$ and none for $\lambda \in (0, \lambda^*]$.

(f) λ_* and λ^* are strictly decreasing while λ^{**} is strictly increasing with respect to L .

Theorem 2.19 (Type IV- γ_1 , $f'(0) > 0$, see Fig.5). Let (φ, f) be of Type IV- γ_1 . Assume conditions (1.2)–(1.6) and $f'(0) > 0$ hold. Then there exist constants $L^{**} > L^* > L_* > 0$ (see Fig.2) such that the following assertions hold:

(a) If $L \geq L^{**}$, then (1.1) has exactly one positive solution for $\lambda \in (0, \lambda_1)$ and none for $\lambda \in [\lambda_1, +\infty)$.

(b) If $L^* < L < L^{**}$, then there exists $\lambda_* \in (0, \lambda_1)$ such that (1.1) has exactly one positive solution for (λ_*, λ_1) and none for $\lambda \in (0, \lambda_*] \cup [\lambda_1, +\infty)$.

(c) If $L = L^*$, then there exist two numbers $0 < \lambda_* < \lambda^* < \lambda_1$ such that (1.1) has exactly one positive solution for $\lambda \in (\lambda_*, \lambda^*) \cup (\lambda^*, \lambda_1)$ and none for $\lambda \in (0, \lambda_*] \cup \{\lambda^*\} \cup [\lambda_1, +\infty)$.

(d) If $L_* < L < L^*$, three numbers $0 < \lambda_* < \lambda^{**} < \lambda^* < \lambda_1$ such that (1.1) has exactly one positive solution for $\lambda \in (\lambda_*, \lambda^{**}) \cup (\lambda^*, \lambda_1)$ and none for $\lambda \in (0, \lambda_*] \cup [\lambda^{**}, \lambda^*] \cup [\lambda_1, +\infty)$.

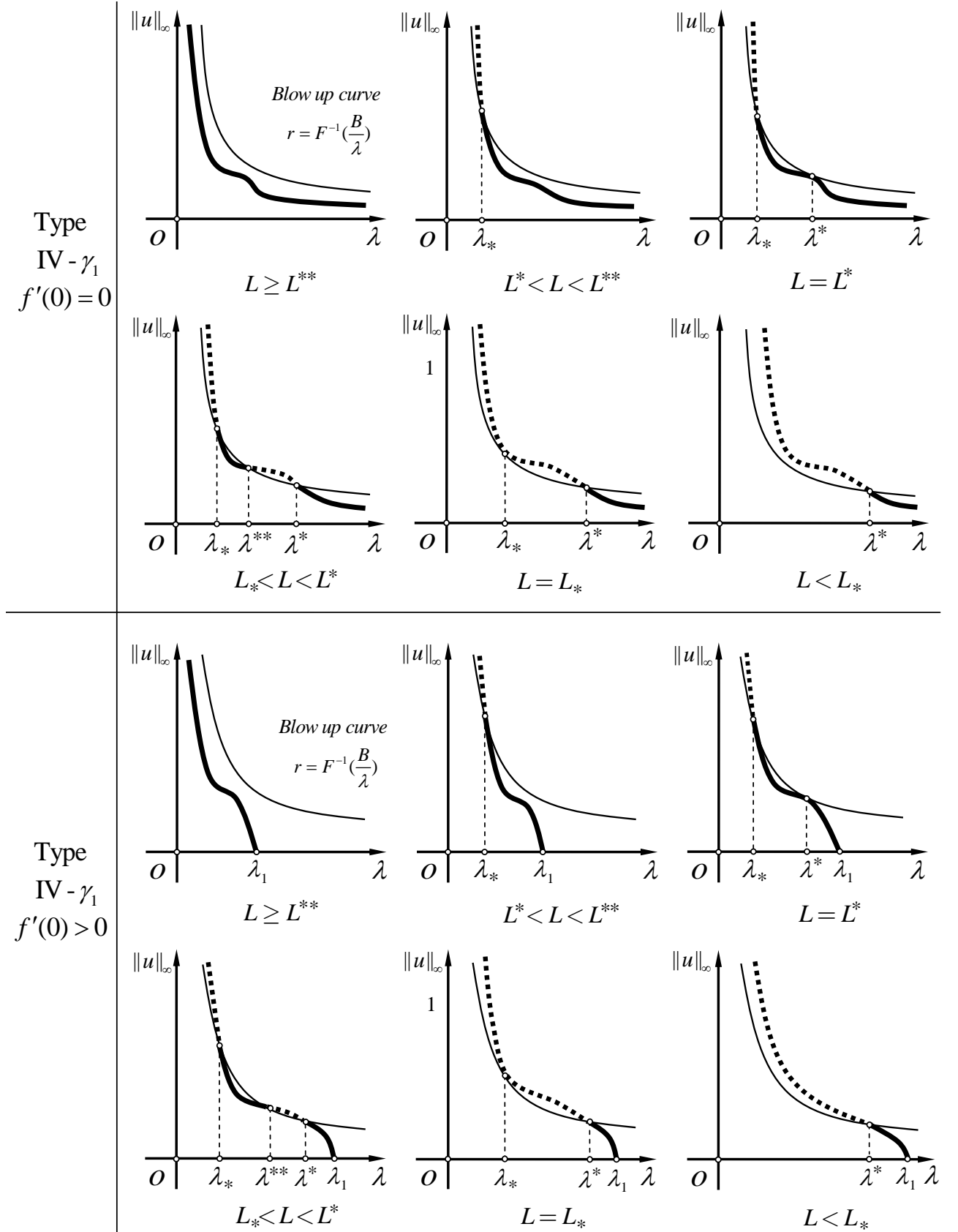
(e) If $L \leq L_*$, then there exists $\lambda^* \in (0, \lambda_1)$ such that (1.1) has exactly one positive solution for (λ^*, λ_1) and none for $\lambda \in (0, \lambda^*] \cup [\lambda_1, +\infty)$.

(f) λ_* and λ^* are strictly decreasing while λ^{**} is strictly increasing with respect to L .

Example 2.15. Let $\varphi = \varphi_3$ and $f(u) = u^7 e^u + u^2$ or $f(u) = u^5 e^u + u$. Then $\lim_{\lambda \rightarrow +\infty} g(\lambda) = 0$ and $\lim_{\lambda \rightarrow 0} g(\lambda) = \frac{\pi}{2}$ (by Lemma 3.9 below). Numerical simulation indicates that $g(\lambda)$ has exactly two local extreme points in $(0, +\infty)$ and $\lim_{\lambda \rightarrow 0} g(\lambda)$ is greater than the local maximum value (see Remark 3.1 and Fig.15 below), which suggests that g is of Type γ_1 and both $(\varphi_3, u^7 e^u + u^2)$ and $(\varphi_3, u^5 e^u + u)$ are of Type IV- γ_1 .

Theorem 2.20 (Type IV- γ_2 , $f'(0) = 0$, see Fig.6). Let (φ, f) be of Type IV- γ_2 . Assume conditions (1.2)–(1.6) and $f'(0) = 0$ hold. Then there exist constants $L^{**} > L^* > L_* > 0$ (see Fig.2) such that the following assertions hold:

(a) If $L > L^{**}$, then (1.1) has exactly one positive solution for any $\lambda \in (0, +\infty)$.

FIGURE 5. Bifurcation Diagrams for Type IV- γ_1 with $f(0) = 0$.

- (b) If $L = L^{**}$, then there exists $\lambda^* > 0$ such that (1.1) has exactly one positive solution for $(0, \lambda^*) \cup (\lambda^*, +\infty)$ and none for $\lambda = \lambda^*$.
- (c) If $L^* \leq L < L^{**}$, then there exist two numbers $\lambda^* > \lambda^{**} > 0$ such that (1.1) has exactly one positive solutions for $\lambda \in (0, \lambda^{**}) \cup (\lambda^*, +\infty)$ and none for $\lambda \in [\lambda^{**}, \lambda^*]$.
- (d) If $L_* < L < L^*$, then there exist three numbers $\lambda^* > \lambda^{**} > \lambda_* > 0$ such that (1.1) has exactly one positive solutions for $\lambda \in (\lambda_*, \lambda^{**}) \cup (\lambda^*, +\infty)$ and none for $\lambda \in (0, \lambda_*] \cup [\lambda^{**}, \lambda^*]$.
- (e) If $L \leq L_*$, then there exists $\lambda^* > 0$ such that (1.1) has exactly one positive solutions for $\lambda \in (\lambda^*, +\infty)$ and none for $\lambda \in (0, \lambda^*]$.
- (f) λ_* and λ^* are strictly decreasing while λ^{**} is strictly increasing with respect to L .

Theorem 2.21 (Type IV- γ_2 , $f'(0) > 0$, see Fig.6). Let (φ, f) be of Type IV- γ_2 . Assume conditions (1.2)–(1.6) and $f'(0) > 0$ hold. Then there exist constants $L^{**} > L^* > L_* > 0$ (see Fig.2) such that the following assertions hold:

- (a) If $L > L^{**}$, then (1.1) has exactly one positive solution for $\lambda \in (0, \lambda_1)$ and none for $\lambda \in [\lambda_1, +\infty)$.
- (b) If $L = L^{**}$, then there exists $\lambda^* \in (0, \lambda_1)$ such that (1.1) has exactly one positive solution for $(0, \lambda^*) \cup (\lambda^*, \lambda_1)$ and none for $\lambda \in \{\lambda^*\} \cup [\lambda_1, +\infty)$.
- (c) If $L^* \leq L < L^{**}$, then there exist two numbers $0 < \lambda^{**} < \lambda^* < \lambda_1$ such that (1.1) has exactly one positive solution for $\lambda \in (0, \lambda^{**}) \cup (\lambda^*, \lambda_1)$ and none for $\lambda \in [\lambda^{**}, \lambda^*] \cup [\lambda_1, +\infty)$.
- (d) If $L_* < L < L^*$, then there exist three numbers $0 < \lambda_* < \lambda^{**} < \lambda^* < \lambda_1$ such that (1.1) has exactly one positive solutions for $\lambda \in (\lambda_*, \lambda^{**}) \cup (\lambda^*, \lambda_1)$ and none for $\lambda \in (0, \lambda_*] \cup [\lambda^{**}, \lambda^*] \cup [\lambda_1, +\infty)$.
- (e) If $L \leq L_*$, then there exists $\lambda^* \in (0, \lambda_1)$ such that (1.1) has exactly one positive solutions for $\lambda \in (\lambda^*, \lambda_1)$ and none for $\lambda \in (0, \lambda^*] \cup [\lambda_1, +\infty)$.
- (f) λ_* and λ^* are strictly decreasing while λ^{**} is strictly increasing with respect to L .

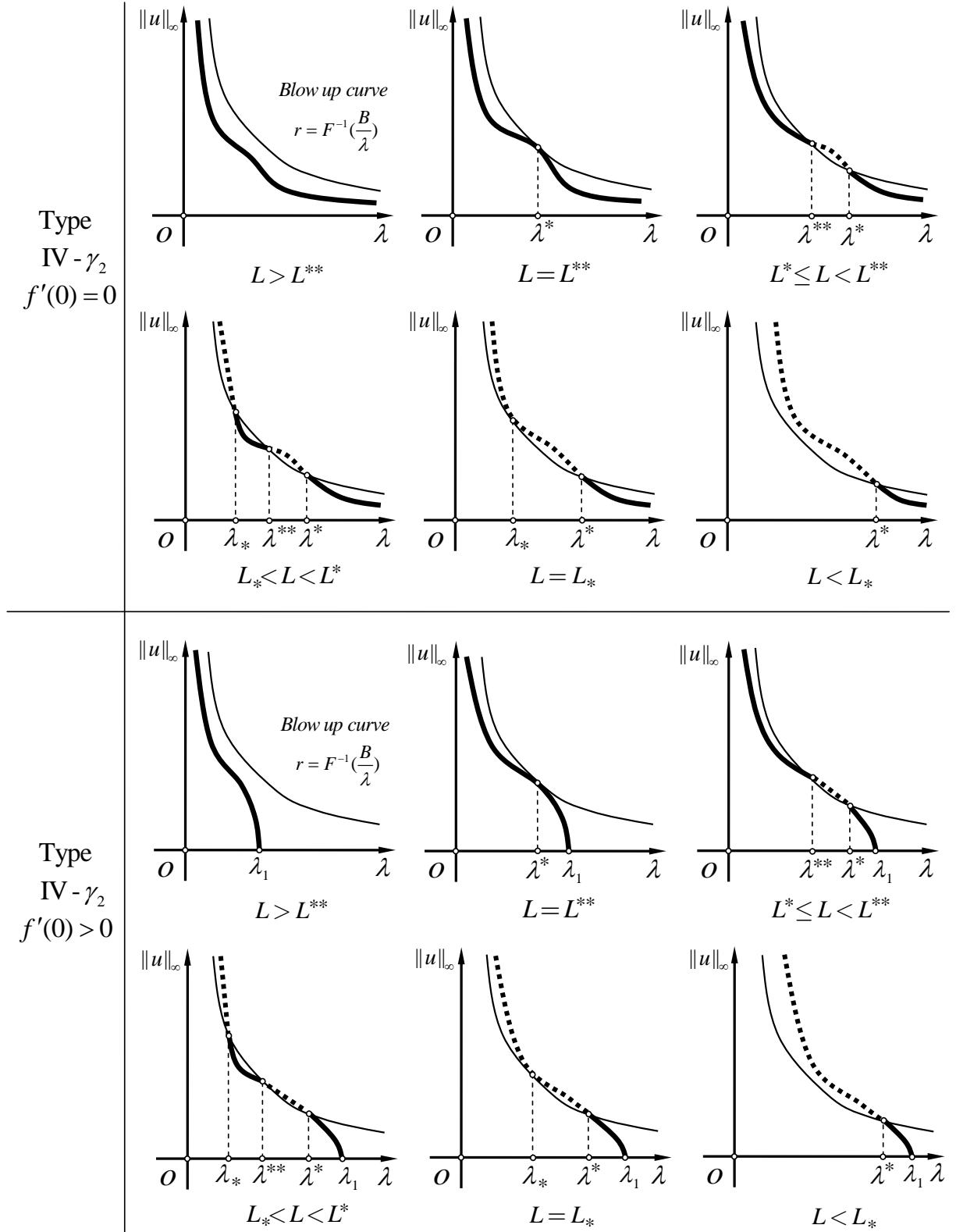
Example 2.16. Let $\varphi = \varphi_3$ and $f(u) = u^7 e^{12u} + u^2$ or $f(u) = u^5 e^{8u} + u$. Then $\lim_{\lambda \rightarrow +\infty} g(\lambda) = 0$ and $\lim_{\lambda \rightarrow 0} g(\lambda) = \frac{\pi}{24}$ or $\frac{\pi}{16}$ (by Lemma 3.9 below). Numerical simulation indicates that $g(\lambda)$ has exactly two local extreme points in $(0, +\infty)$ and $\lim_{\lambda \rightarrow 0} g(\lambda)$ is in between the local extreme values (see Remark 3.1 and Fig.15 below), which suggests that g is of Type γ_1 and both $(\varphi_3, u^7 e^{12u} + u^2)$ and $(\varphi_3, u^5 e^{8u} + u)$ are of Type IV- γ_2 .

Theorem 2.22 (Type IV- γ_3 , $f'(0) = 0$, see Fig.7). Let (φ, f) be of Type IV- γ_3 . Assume conditions (1.2)–(1.6) and $f'(0) = 0$ hold. Then there exist constants $L^* > L_* > 0$ (see Fig.2) such that the following assertions hold:

- (a) If $L > L^*$, then (1.1) has exactly one positive solution for any $\lambda \in (0, +\infty)$.
- (b) If $L = L^*$, then there exists $\lambda^* > 0$ such that (1.1) has exactly one positive solution for $(0, \lambda^*) \cup (\lambda^*, +\infty)$ and none for $\lambda = \lambda^*$.
- (c) If $L_* < L < L^*$, then there exist three numbers $\lambda^* > \lambda^{**} > \lambda_* > 0$ such that (1.1) has exactly one positive solutions for $\lambda \in (\lambda_*, \lambda^{**}) \cup (\lambda^*, +\infty)$ and none for $\lambda \in (0, \lambda_*] \cup [\lambda^{**}, \lambda^*]$.
- (d) If $L \leq L_*$, then there exists $\lambda^* > 0$ such that (1.1) has exactly one positive solutions for $\lambda \in (\lambda^*, +\infty)$ and none for $\lambda \in (0, \lambda^*]$.
- (e) λ_* and λ^* are strictly decreasing while λ^{**} is strictly increasing with respect to L .

Theorem 2.23 (Type IV- γ_3 , $f'(0) > 0$, see Fig.8). Let (φ, f) be of Type IV- γ_3 . Assume conditions (1.2)–(1.6) and $f'(0) > 0$ hold. Then there exist constants $L^* > L_* > 0$ (see Fig.2) such that the following assertions hold:

- (a) If $L > L^*$, then (1.1) has exactly one positive solution for $\lambda \in (0, \lambda_1)$ and none for $\lambda \in [\lambda_1, +\infty)$.
- (b) If $L = L^*$, then there exists $\lambda^* \in (0, \lambda_1)$ such that (1.1) has exactly one positive solution for $(0, \lambda^*) \cup (\lambda^*, \lambda_1)$ and none for $\lambda \in \{\lambda^*\} \cup [\lambda_1, +\infty)$.

FIGURE 6. Bifurcation Diagrams for Type IV- γ_2 with $f(0) = 0$.

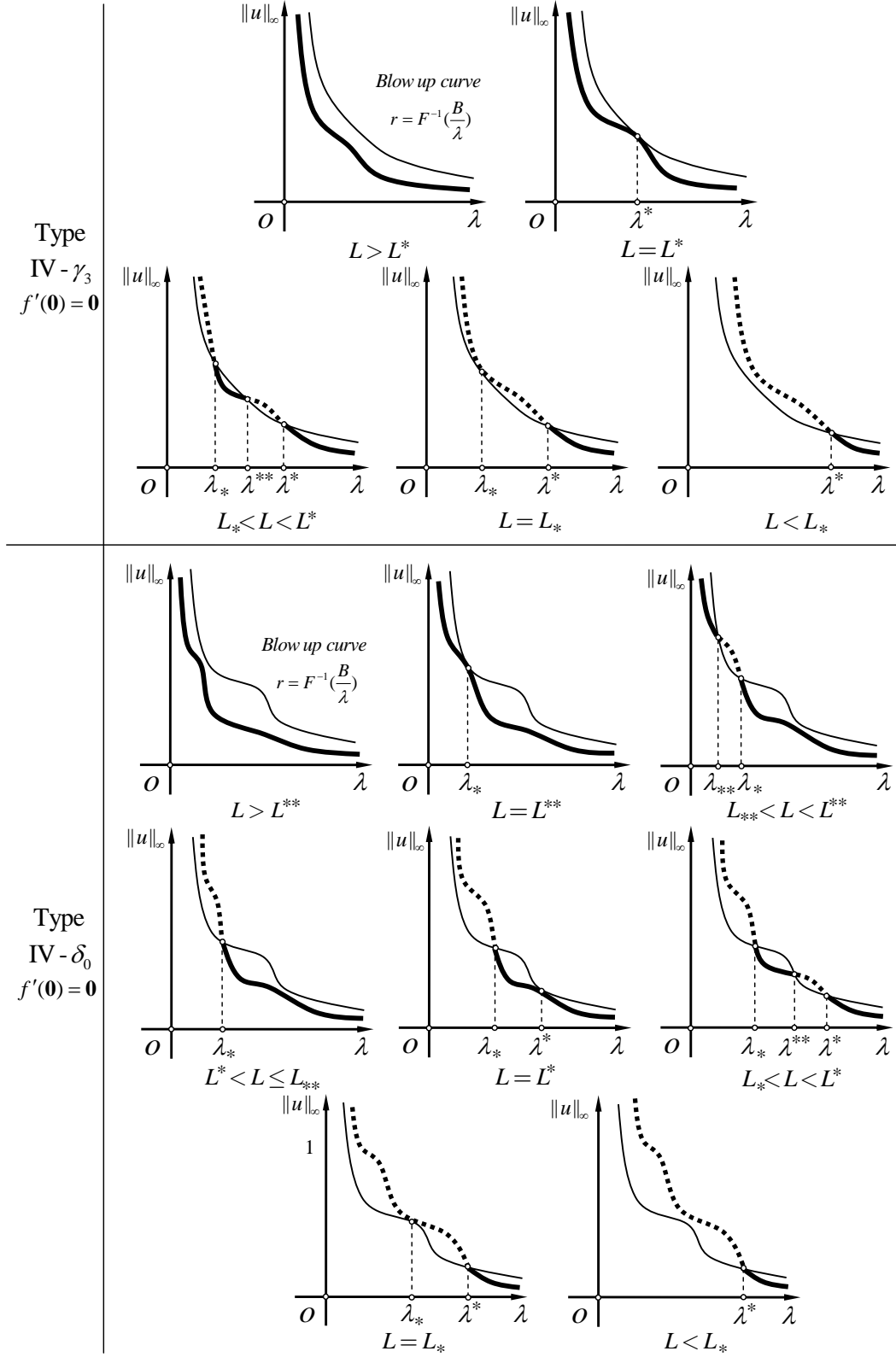


FIGURE 7. Bifurcation Diagrams for Types IV- γ_3 and IV- δ_0 with $f(0) = 0$ and $f'(0) = 0$.

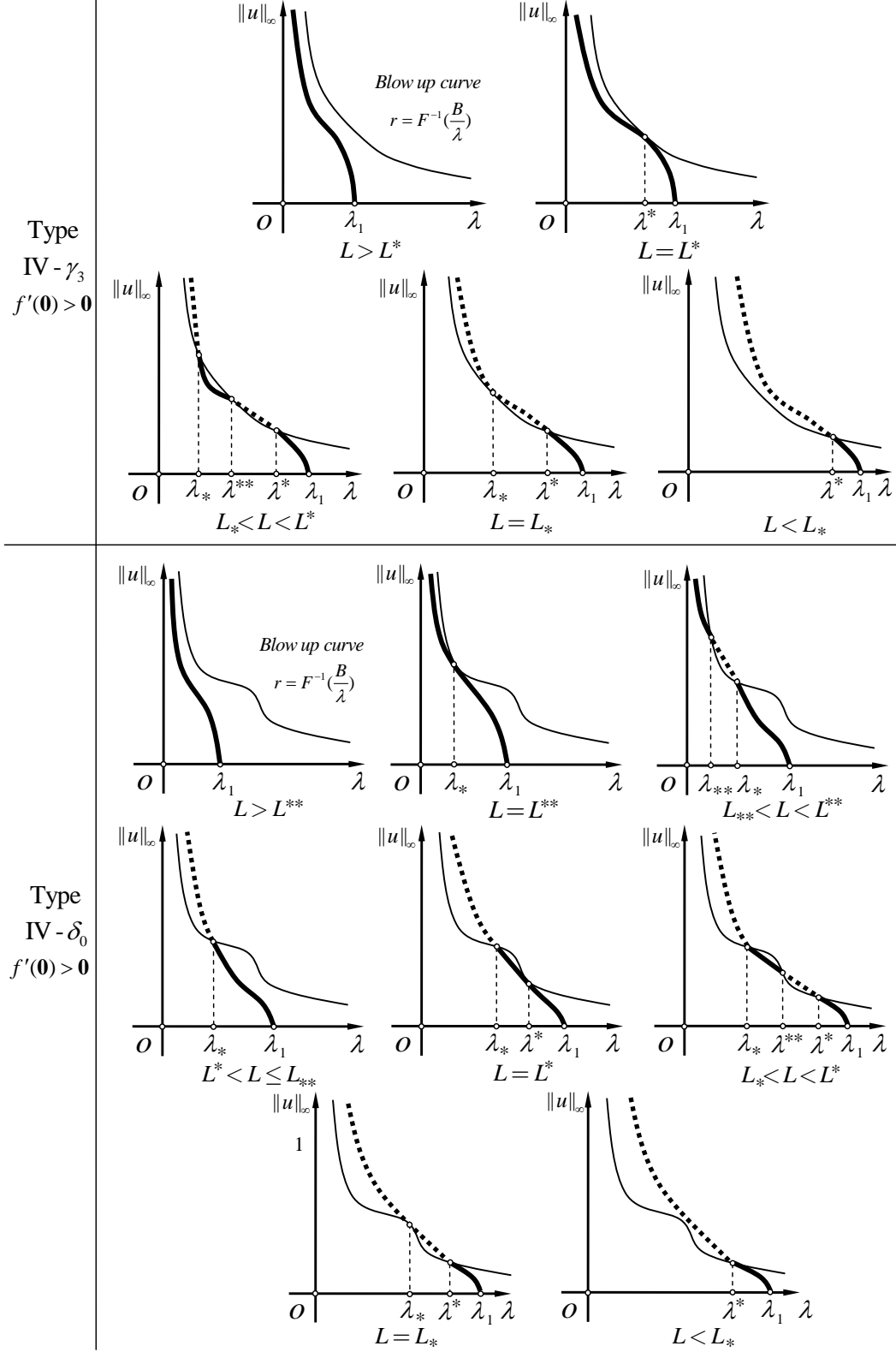


FIGURE 8. Bifurcation Diagrams for Types IV- γ_3 and IV- δ_0 with $f(0) = 0$ and $f'(0) > 0$.

(c) If $L_* < L < L^*$, then there exist three numbers $0 < \lambda_* < \lambda^{**} < \lambda^* < \lambda_1$ such that (1.1) has exactly one positive solutions for $\lambda \in (\lambda_*, \lambda^{**}) \cup (\lambda^*, \lambda_1)$ and none for $\lambda \in (0, \lambda_*) \cup [\lambda^{**}, \lambda^*] \cup [\lambda_1, +\infty)$.

(d) If $L \leq L_*$, then there exists $\lambda^* \in (0, \lambda_1)$ such that (1.1) has exactly one positive solutions for $\lambda \in (\lambda^*, \lambda_1)$ and none for $\lambda \in (0, \lambda^*] \cup [\lambda_1, +\infty)$.

(e) λ_* and λ^* are strictly decreasing while λ^{**} is strictly increasing with respect to L .

Example 2.17. Let $\varphi = \varphi_3$. By Lemma 3.9 and a comparison between Examples 2.15 and 2.16 (also see Figs. 14 and 15), we conjecture that there exist $k_1 \in (1, 12)$ and $k_2 \in (1, 8)$ such that $(\varphi_3, u^7 e^{k_1 u} + u^2)$ and $(\varphi_3, u^5 e^{k_2 u} + u)$ are of Type IV- γ_3 , i.e., g has exactly two local extreme points and the local maximum value is equal to $\lim_{\lambda \rightarrow 0} g(\lambda)$.

Theorem 2.24 (Type IV- δ_0 , $f'(0) = 0$, see Fig. 7). Let (φ, f) be of Type IV- δ_0 . Assume conditions (1.2)–(1.6) and $f'(0) = 0$ hold. Then there exist constants $L^{**} > L_{**} > L^* > L_* > 0$ (see Fig. 2) such that the following assertions hold:

(a) If $L > L^{**}$, then (1.1) has exactly one positive solution for any $\lambda \in (0, +\infty)$.

(b) If $L = L^{**}$, then there exists $\lambda_* > 0$ such that (1.1) has exactly one positive solution for $(0, \lambda_*) \cup (\lambda_*, +\infty)$ and none for $\lambda = \lambda_*$.

(c) If $L_{**} < L < L^{**}$, then there exist two numbers $\lambda_* > \lambda_{**} > 0$ such that (1.1) has exactly one positive solutions for $\lambda \in (0, \lambda_{**}) \cup (\lambda_*, +\infty)$ and none for $\lambda \in [\lambda_{**}, \lambda_*]$.

(d) If $L^* < L \leq L_{**}$, then there exists $\lambda_* > 0$ such that (1.1) has exactly one positive solutions for $\lambda \in (\lambda_*, +\infty)$ and none for $\lambda \in (0, \lambda_*]$.

(e) If $L = L^*$, then there exist two numbers $\lambda^* > \lambda_* > 0$ such that (1.1) has exactly one positive solutions for $\lambda \in (\lambda_*, \lambda^*) \cup (\lambda^*, +\infty)$ and none for $\lambda \in (0, \lambda_*] \cup \{\lambda^*\}$.

(f) If $L_* < L < L^*$, then there exist three numbers $\lambda^* > \lambda^{**} > \lambda_* > 0$ such that (1.1) has exactly one positive solutions for $\lambda \in (\lambda_*, \lambda^{**}) \cup (\lambda^*, +\infty)$ and none for $\lambda \in (0, \lambda_*) \cup [\lambda^{**}, \lambda^*]$.

(g) If $L \leq L_*$, then there exists $\lambda^* > 0$ such that (1.1) has exactly one positive solutions for $\lambda \in (\lambda^*, +\infty)$ and none for $\lambda \in (0, \lambda^*]$.

(h) λ_* and λ^* are strictly decreasing while λ_{**} and λ^{**} are strictly increasing with respect to L .

Theorem 2.25 (Type IV- δ_0 , $f'(0) > 0$, see Fig. 8). Let (φ, f) be of Type IV- δ_0 . Assume conditions (1.2)–(1.6) and $f'(0) > 0$ hold. Then there exist constants $L^{**} > L_{**} > L^* > L_* > 0$ (see Fig. 2) such that the following assertions hold:

(a) If $L > L^{**}$, then (1.1) has exactly one positive solution for $\lambda \in (0, \lambda_1)$ and none for $\lambda \in [\lambda_1, +\infty)$.

(b) If $L = L^{**}$, then there exists $\lambda_* \in (0, \lambda_1)$ such that (1.1) has exactly one positive solution for $(0, \lambda_*) \cup (\lambda_*, \lambda_1)$ and none for $\lambda \in \{\lambda_*\} \cup [\lambda_1, +\infty)$.

(c) If $L_{**} < L < L^{**}$, then there exist two numbers $0 < \lambda_{**} < \lambda_* < \lambda_1$ such that (1.1) has exactly one positive solution for $\lambda \in (0, \lambda_{**}) \cup (\lambda_*, \lambda_1)$ and none for $\lambda \in [\lambda_{**}, \lambda_*] \cup [\lambda_1, +\infty)$.

(d) If $L^* < L \leq L_{**}$, then there exists $\lambda_* \in (0, \lambda_1)$ such that (1.1) has exactly one positive solutions for $\lambda \in (\lambda_*, \lambda_1)$ and none for $\lambda \in (0, \lambda_*] \cup [\lambda_1, +\infty)$.

(e) If $L = L^*$, then there exist two numbers $0 < \lambda_* < \lambda^* < \lambda_1$ such that (1.1) has exactly one positive solutions for $\lambda \in (\lambda_*, \lambda^*) \cup (\lambda^*, \lambda_1)$ and none for $\lambda \in (0, \lambda_*] \cup \{\lambda^*\} \cup [\lambda_1, +\infty)$.

(f) If $L_* < L < L^*$, then there exist three numbers $0 < \lambda_* < \lambda^{**} < \lambda^* < \lambda_1$ such that (1.1) has exactly one positive solutions for $\lambda \in (\lambda_*, \lambda^{**}) \cup (\lambda^*, \lambda_1)$ and none for $\lambda \in (0, \lambda_*) \cup [\lambda^{**}, \lambda^*] \cup [\lambda_1, +\infty)$.

(g) If $L \leq L_*$, then there exists $\lambda^* \in (0, \lambda_1)$ such that (1.1) has exactly one positive solutions for $\lambda \in (\lambda^*, \lambda_1)$ and none for $\lambda \in (0, \lambda^*] \cup [\lambda_1, +\infty)$.

(h) λ_* and λ^* are strictly decreasing while λ^{**} are strictly increasing with respect to L .

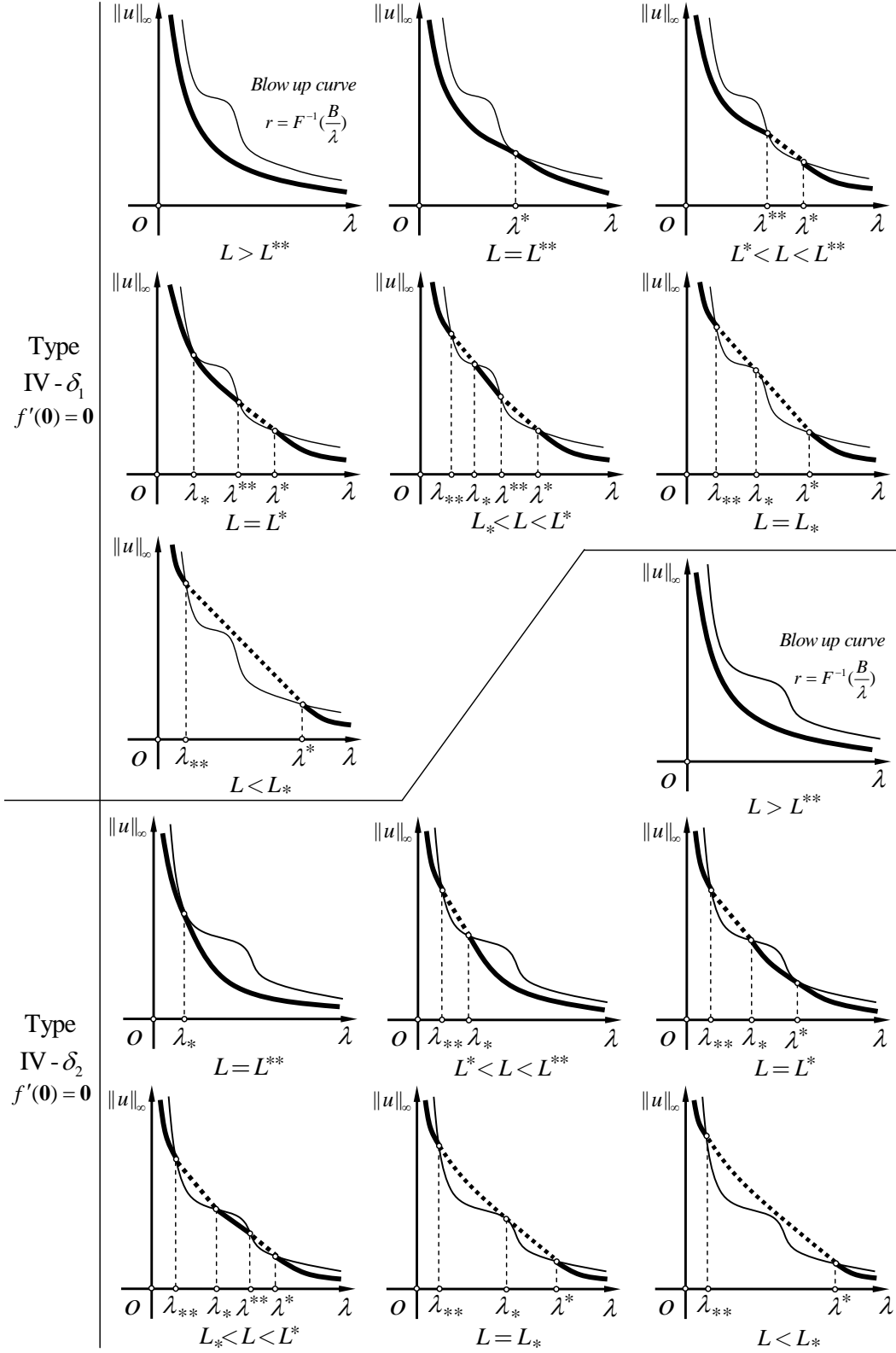


FIGURE 9. Bifurcation Diagrams for Types IV- δ_1 and IV- δ_2 with $f(0) = 0$ and $f'(0) = 0$.

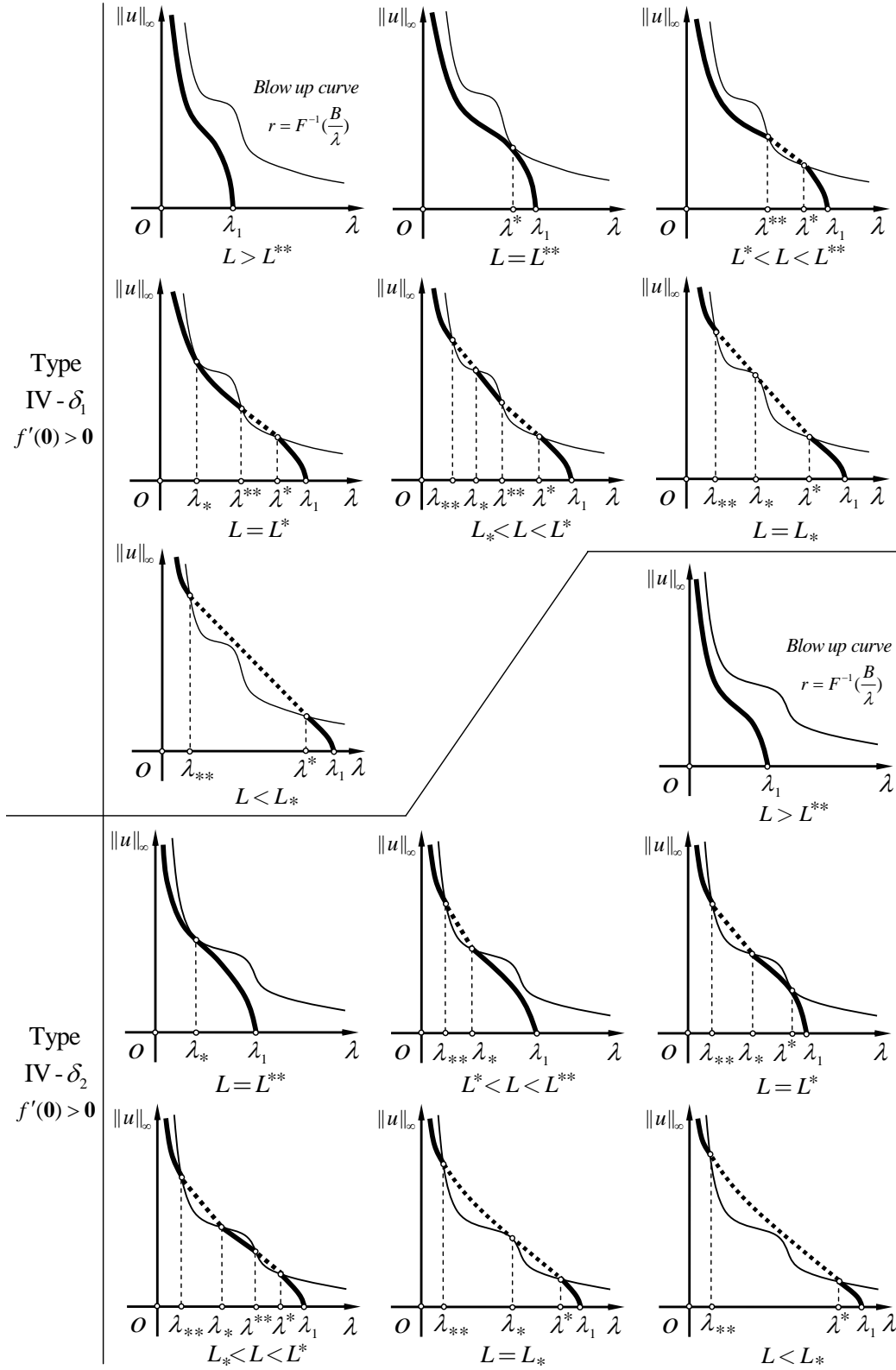


FIGURE 10. Bifurcation Diagrams for Types IV- δ_1 and IV- δ_2 with $f(0) = 0$ and $f'(0) > 0$.

Example 2.18. Let $\varphi = \varphi_3$ and $f(u) = e^u + u^8 - u - 1$ or $f(u) = e^u + u^8 - 1$. Then $\lim_{\lambda \rightarrow +\infty} g(\lambda) = 0$ and $\lim_{\lambda \rightarrow 0} g(\lambda) = \frac{\pi}{2}$ (by Lemma 3.9 below). Numerical simulation indicates that $g(\lambda)$ has exactly three local extreme points in $(0, +\infty)$ and $\lim_{\lambda \rightarrow 0} g(\lambda)$ is in between the local maximum values (see Remark 3.1 and Fig.15 below), which suggests that g is of Type δ_0 and both $(\varphi_3, e^u + u^8 - u - 1)$ and $(\varphi_3, e^u + u^8 - 1)$ are of Type IV- δ_0 .

Theorem 2.26 (Type IV- δ_1 , $f'(0) = 0$, see Fig.9). Let (φ, f) be of Type IV- δ_1 . Assume conditions (1.2)–(1.6) and $f'(0) = 0$ hold. Then there exist constants $L^{**} > L^* > L_* > 0$ (see Fig.2) such that the following assertions hold:

- (a) If $L > L^{**}$, then (1.1) has exactly one positive solution for any $\lambda \in (0, +\infty)$.
- (b) If $L = L^{**}$, then there exists $\lambda^* > 0$ such that (1.1) has exactly one positive solution for $(0, \lambda^*) \cup (\lambda^*, +\infty)$ and none for $\lambda = \lambda^*$.
- (c) If $L^* < L < L^{**}$, then there exist two numbers $\lambda^* > \lambda^{**} > 0$ such that (1.1) has exactly one positive solutions for $\lambda \in (0, \lambda^{**}) \cup (\lambda^*, +\infty)$ and none for $\lambda \in [\lambda^{**}, \lambda^*]$.
- (d) If $L = L^*$, then there exist three numbers $\lambda^* > \lambda^{**} > \lambda_* > 0$ such that (1.1) has exactly one positive solutions for $\lambda \in (0, \lambda_*) \cup (\lambda_*, \lambda^{**}) \cup (\lambda^*, +\infty)$ and none for $\lambda \in \{\lambda_*\} \cup [\lambda^{**}, \lambda^*]$.
- (e) If $L_* < L < L^*$, then there exist four numbers $\lambda^* > \lambda^{**} > \lambda_* > \lambda_{**} > 0$ such that (1.1) has exactly one positive solutions for $\lambda \in (0, \lambda_{**}) \cup (\lambda_*, \lambda^{**}) \cup (\lambda^*, +\infty)$ and none for $\lambda \in [\lambda_{**}, \lambda_*] \cup [\lambda^{**}, \lambda^*]$.
- (f) If $L \leq L_*$, then there exist two numbers $\lambda^* > \lambda_{**} > 0$ such that (1.1) has exactly one positive solutions for $\lambda \in (0, \lambda_{**}) \cup (\lambda^*, +\infty)$ and none for $\lambda \in [\lambda_{**}, \lambda^*]$.
- (g) λ_* and λ^* are strictly decreasing while λ_{**} and λ^{**} are strictly increasing with respect to L .

Theorem 2.27 (Type IV- δ_1 , $f'(0) > 0$, see Fig.10). Let (φ, f) be of Type IV- δ_1 . Assume conditions (1.2)–(1.6) and $f'(0) > 0$ hold. Then there exist constants $L^{**} > L^* > L_* > 0$ (see Fig.2) such that the following assertions hold:

- (a) If $L > L^{**}$, then (1.1) has exactly one positive solution for $\lambda \in (0, \lambda_1)$ and none for $\lambda \in [\lambda_1, +\infty)$.
- (b) If $L = L^{**}$, then there exists $\lambda^* \in (0, \lambda_1)$ such that (1.1) has exactly one positive solution for $(0, \lambda^*) \cup (\lambda^*, \lambda_1)$ and none for $\lambda \in \{\lambda^*\} \cup [\lambda_1, +\infty)$.
- (c) If $L^* < L < L^{**}$, then there exist two numbers $0 < \lambda^{**} < \lambda^* < \lambda_1$ such that (1.1) has exactly one positive solution for $\lambda \in (0, \lambda^{**}) \cup (\lambda^*, \lambda_1)$ and none for $\lambda \in [\lambda^{**}, \lambda^*] \cup [\lambda_1, +\infty)$.
- (d) If $L = L^*$, then there exist three numbers $0 < \lambda_* < \lambda^{**} < \lambda^* < \lambda_1$ such that (1.1) has exactly one positive solutions for $\lambda \in (0, \lambda_*) \cup (\lambda_*, \lambda^{**}) \cup (\lambda^*, \lambda_1)$ and none for $\lambda \in \{\lambda_*\} \cup [\lambda_{**}, \lambda_*] \cup [\lambda_1, +\infty)$.
- (e) If $L_* < L < L^*$, then there exist four numbers $0 < \lambda_{**} < \lambda_* < \lambda^{**} < \lambda^* < \lambda_1$ such that (1.1) has exactly one positive solutions for $\lambda \in (0, \lambda_{**}) \cup (\lambda_*, \lambda^{**}) \cup (\lambda^*, \lambda_1)$ and none for $\lambda \in [\lambda_{**}, \lambda_*] \cup [\lambda^{**}, \lambda^*] \cup [\lambda_1, +\infty)$.
- (f) If $L \leq L_*$, then there exist two numbers two numbers $0 < \lambda_{**} < \lambda^* < \lambda_1$ such that (1.1) has exactly one positive solution for $\lambda \in (0, \lambda_{**}) \cup (\lambda^*, \lambda_1)$ and none for $\lambda \in [\lambda_{**}, \lambda^*] \cup [\lambda_1, +\infty)$.
- (g) λ_* and λ^* are strictly decreasing while λ_{**} and λ^{**} are strictly increasing with respect to L .

Example 2.19. Let $\varphi = \varphi_3$ and $f(u) = e^{u^2} + u^8 + u - 1$. Then $\lim_{\lambda \rightarrow +\infty} g(\lambda) = 0 = \lim_{\lambda \rightarrow 0} g(\lambda)$ (by Lemma 3.9 below). Numerical simulation indicates that $g(\lambda)$ has exactly three local extreme points in $(0, +\infty)$ and the left local maximum value is less than the right one (see Remark 3.1 and Fig.16 below), which suggests that g is of Type δ_1 and $(\varphi_3, e^{u^2} + u^8 + u - 1)$ is of Type IV- δ_1 .

Theorem 2.28 (Type IV- δ_2 , $f'(0) = 0$, see Fig.9). Let (φ, f) be of Type IV- δ_2 . Assume conditions (1.2)–(1.6) and $f'(0) = 0$ hold. Then there exist constants $L^{**} > L^* > L_* > 0$ (see Fig.2) such that the following assertions hold:

- (a) If $L > L^{**}$, then (1.1) has exactly one positive solution for any $\lambda \in (0, +\infty)$.

- (b) If $L = L^{**}$, then there exists $\lambda_* > 0$ such that (1.1) has exactly one positive solution for $(0, \lambda_*) \cup (\lambda_*, +\infty)$ and none for $\lambda = \lambda_*$.
- (c) If $L^* < L < L^{**}$, then there exist two numbers $\lambda_* > \lambda_{**} > 0$ such that (1.1) has exactly one positive solutions for $\lambda \in (0, \lambda_{**}) \cup (\lambda_*, +\infty)$ and none for $\lambda \in [\lambda_{**}, \lambda_*]$.
- (d) If $L = L^*$, then there exist three numbers $\lambda^* > \lambda_* > \lambda_{**} > 0$ such that (1.1) has exactly one positive solutions for $\lambda \in (0, \lambda_{**}) \cup (\lambda_*, \lambda^*) \cup (\lambda^*, +\infty)$ and none for $\lambda \in [\lambda_{**}, \lambda_*] \cup \{\lambda^*\}$.
- (e) If $L_* < L < L^*$, then there exist five numbers $\lambda^* > \lambda^{**} > \lambda_* > \lambda_{**} > 0$ such that (1.1) has exactly one positive solutions for $\lambda \in (0, \lambda_{**}) \cup (\lambda_*, \lambda^{**}) \cup (\lambda^*, +\infty)$ and none for $\lambda \in [\lambda_{**}, \lambda_*] \cup [\lambda^{**}, \lambda^*]$.
- (f) If $L \leq L_*$, then there exist two numbers $\lambda^* > \lambda_{**} > 0$ such that (1.1) has exactly one positive solutions for $\lambda \in (0, \lambda_{**}) \cup (\lambda^*, +\infty)$ and none for $\lambda \in [\lambda_{**}, \lambda^*]$.
- (g) λ_* and λ^* are strictly decreasing while λ_{**} and λ^{**} are strictly increasing with respect to L .

Theorem 2.29 (Type IV- δ_2 , $f'(0) > 0$, see Fig.10). Let (φ, f) be of Type IV- δ_2 . Assume conditions (1.2)–(1.6) and $f'(0) > 0$ hold. Then there exist constants $L^{**} > L^* > L_* > 0$ (see Fig.2) such that the following assertions hold:

- (a) If $L > L^{**}$, then (1.1) has exactly one positive solution for $\lambda \in (0, \lambda_1)$ and none for $\lambda \in [\lambda_1, +\infty)$.
- (b) If $L = L^{**}$, then there exists $\lambda_* \in (0, \lambda_1)$ such that (1.1) has exactly one positive solution for $(0, \lambda_*) \cup (\lambda_*, \lambda_1)$ and none for $\lambda \in \{\lambda_*\} \cup [\lambda_1, +\infty)$.
- (c) If $L^* < L < L^{**}$, then there exist two numbers $0 < \lambda_{**} < \lambda_* < \lambda_1$ such that (1.1) has exactly one positive solution for $\lambda \in (0, \lambda_{**}) \cup (\lambda_*, \lambda_1)$ and none for $\lambda \in [\lambda_{**}, \lambda_*] \cup [\lambda_1, +\infty)$.
- (d) If $L = L^*$, then there exist three numbers $0 < \lambda_{**} < \lambda_* < \lambda^* < \lambda_1$ such that (1.1) has exactly one positive solutions for $\lambda \in (0, \lambda_{**}) \cup (\lambda_*, \lambda^*) \cup (\lambda^*, \lambda_1)$ and none for $\lambda \in [\lambda_{**}, \lambda_*] \cup \{\lambda^*\} \cup [\lambda_1, +\infty)$.
- (e) If $L_* < L < L^*$, then there exist four numbers $0 < \lambda_{**} < \lambda_* < \lambda^{**} < \lambda^* < \lambda_1$ such that (1.1) has exactly one positive solutions for $\lambda \in (0, \lambda_{**}) \cup (\lambda_*, \lambda^{**}) \cup (\lambda^*, \lambda_1)$ and none for $\lambda \in [\lambda_{**}, \lambda_*] \cup [\lambda^{**}, \lambda^*] \cup [\lambda_1, +\infty)$.
- (f) If $L \leq L_*$, then there exist two numbers $0 < \lambda_{**} < \lambda^* < \lambda_1$ such that (1.1) has exactly one positive solution for $\lambda \in (0, \lambda_{**}) \cup (\lambda^*, \lambda_1)$ and none for $\lambda \in [\lambda_{**}, \lambda^*] \cup [\lambda_1, +\infty)$.
- (g) λ_* and λ^* are strictly decreasing while λ_{**} and λ^{**} are strictly increasing with respect to L .

Example 2.20. Let $\varphi = \varphi_3$ and $f(u) = e^{u^2} + u^8 - 1$. Then $\lim_{\lambda \rightarrow +\infty} g(\lambda) = 0 = \lim_{\lambda \rightarrow 0} g(\lambda)$ (by Lemma 3.9 below). Numerical simulation indicates that $g(\lambda)$ has exactly three local extreme points in $(0, +\infty)$ and the left local maximum value is greater than the right one (see Remark 3.1 and Fig.16 below), which suggests that g is of Type δ_2 and $(\varphi_3, e^{u^2} + u^8 - 1)$ is of Type IV- δ_2 .

Remark 2.21. We conjecture that there exist suitable φ and f such that the corresponding g is of Type δ_3 . Since it is difficult to find some examples, we omit it.

For Cases V and VI, so far, we only know the existence of two types of g : Type β_0 for Case V and Type γ_0 for Case VI (see Fig.2), which have been defined in [13].

Case V: $B < +\infty$, $A < +\infty$ and $C = +\infty$

For Type V- β_0 , the statements of results are the same as Type IV- β_0 (see Theorems 2.12 and 2.13), so we omit them. The bifurcation diagrams are sketched in Figs.11 and 12, which are slightly different from those of Type IV- β_0 (see Figs.3 and 4).

Example 2.22. Let $\varphi = \varphi_3$ and f is one of the following

- (1) $\tan u$, $u \in (0, \frac{\pi}{2})$; (2) $\tan u^2$, $u \in (0, \sqrt{\frac{\pi}{2}})$; (3) $\frac{u}{1-u}$, $u \in (0, 1)$; (4) $\frac{u^2}{1-u^2}$, $u \in (0, 1)$.

Then $\lim_{\lambda \rightarrow +\infty} g(\lambda) = 0 = \lim_{\lambda \rightarrow 0} g(\lambda)$ (by Lemma 3.9 below). Numerical simulation indicates that $g(\lambda)$ has exactly one critical point in $(0, A)$ (see Remark 3.1 and Fig.16 below), which suggests that g is of Type β_0 and all of (φ_3, f) are of Type V- β_0 .

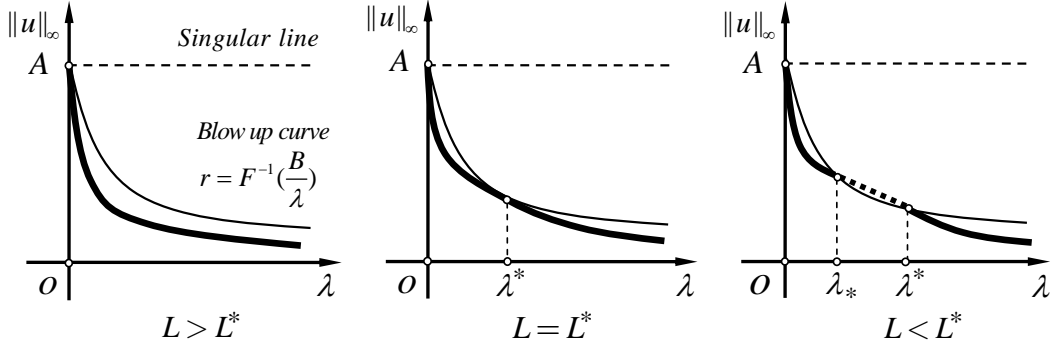


FIGURE 11. Bifurcation Diagrams for Type V- β_0 with $f(0) = 0$ and $f'(0) = 0$.

Case VI: $B < +\infty$, $A < +\infty$ and $C < +\infty$

For Type VI- γ_0 , the statements of results are the same as Type V- γ_0 (see Theorems 2.16 and 2.17), so we omit them. The bifurcation diagrams are sketched in Fig.12, which has some important differences from those of Type IV- γ_0 (see Figs.3 and 4).

Example 2.23. Let $\varphi = \varphi_3$ and $f(u) = \frac{1}{\sqrt{1-u}} - 1$ or $f(u) = \frac{1}{\sqrt{1-u^2}} - 1$ with $u \in (0, 1)$. Then $\lim_{\lambda \rightarrow 0} g(\lambda) = +\infty$, $\lim_{\lambda \rightarrow +\infty} g(\lambda) = 0$, $\lim_{\lambda \rightarrow 1} g(\lambda) > 0$ or $\lim_{\lambda \rightarrow \frac{2}{\pi-2}} g(\lambda) > 0$ (by Lemmas 3.9 and 3.11 below), and $g(\lambda)$ is strictly decreasing in $(0, 1)$ or $(0, \frac{2}{\pi-2})$ (by Lemma 3.12). Moreover, numerical simulation indicates that $g(\lambda)$ has exactly one local maximum point in $(1, +\infty)$ or $(\frac{2}{\pi-2}, +\infty)$ (see Remark 3.1 and Fig.16), which suggests that g is of Type δ_2 and both $(\varphi_3, \frac{1}{\sqrt{1-u}} - 1)$ and $(\varphi_3, \frac{1}{\sqrt{1-u^2}} - 1)$ are of Type VI- γ_0 .

Remark 2.24. In Examples 2.15–2.23, we only give numerical results on the number of local extreme points of g . It is still lack of strictly analytic proofs. Thus we leave some open problems. Moreover, it is also an interesting open problem to find out a complete list of all possible types of g , corresponding to φ_3 and convex, increasing f in Cases IV, V and VI.

Remark 2.25. Although we here focus on classical solutions of (1.1), we also deviate a little from the topics of discussion and gives some additional information on “non-classical solutions” in Figs.1, 3–12. In fact, the thick dashed lines represent discontinuous non-classical solutions (because of limitations of space, about non-classical solutions refer to [15, 16], also see [3, 2, 5, 12]). We also note that the intersection points of the bifurcation curve and the singular line in Figs.1, 11 and 12 represent “singular solutions” (see e.g. [10] for the semilinear case).

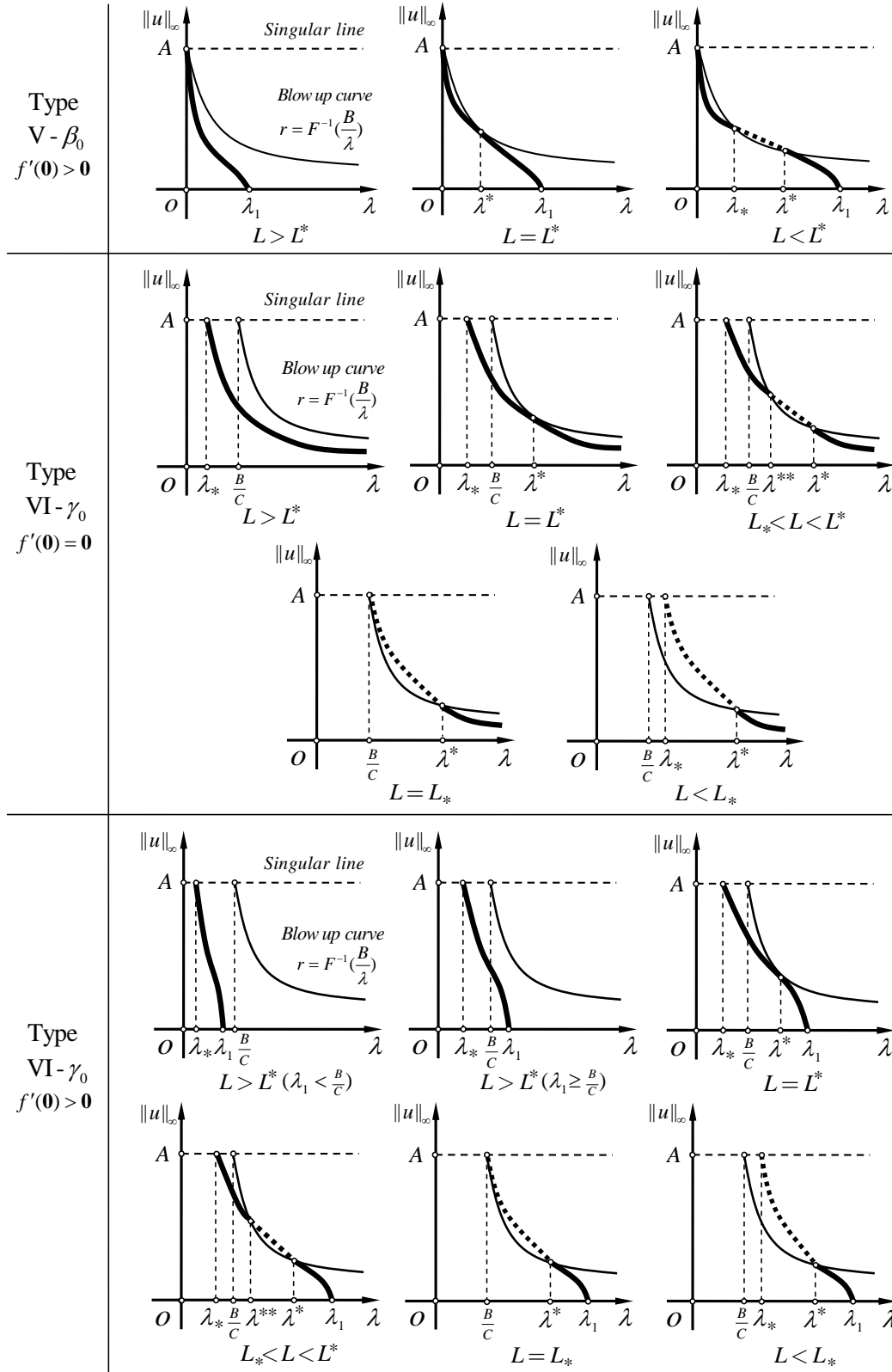
3. TIME MAP AND GENERAL PROPERTIES

In this section, we recall and investigate various properties of the so-called time map T for problem (1.1) under suitable conditions. These properties will be used to obtain the shape of the time map and prove the main theorems.

3.1. Some well known results about time map. Since $f(u)$ does not contain x explicitly and the problem is autonomous, positive solutions of (1.1) are always symmetric (see e.g. [8, Lem 2.1]). Hence $u'(0) = 0$.

Due to the symmetry, we consider the equation

$$-\varphi'(u')u'' = \lambda f(u), \quad x \in (0, L),$$

FIGURE 12. Bifurcation Diagrams for Types V- β_0 and VI- γ_0 with $f(0) = 0$.

with the initial conditions

$$u(0) = r > 0, \quad u'(0) = 0.$$

From the energy conservation relation

$$\Phi(u') + \lambda F(u) = \lambda F(r),$$

we obtain the following time map for positive solutions

$$T(r, \lambda) = \int_0^r \frac{1}{\Phi^{-1}(\lambda [F(r) - F(u)])} du, \quad r \in I, \quad \lambda > 0, \quad (3.1)$$

where Φ^{-1} is always taken to be positive and

$$I := \begin{cases} (0, A), & \text{if } B = +\infty; \\ (0, F^{-1}(\frac{B}{\lambda})], & \text{if } B < +\infty \text{ and } C = +\infty; \\ (0, A), & \text{if } B < +\infty, C < +\infty, \text{ and } \lambda \leq \frac{B}{C}; \\ (0, F^{-1}(\frac{B}{\lambda})], & \text{if } B < +\infty, C < +\infty, \text{ and } \lambda > \frac{B}{C}. \end{cases} \quad (3.2)$$

Denote by r^* the right endpoint of I . Under conditions (1.2) and (1.3), $T(r, \lambda)$ is well defined and continuous for all $(r, \lambda) \in I \times (0, +\infty)$. When $B < +\infty$, the curve $r = F^{-1}(\frac{B}{\lambda})$, due to $|u'(\pm L; r)| = +\infty$, is known as “(gradient or derivative) blow-up curve”; when $A < +\infty$, the straight line $r = A$ is referred as “singular line”. See [13] for details.

Since the solutions of (1.1) correspond to the bifurcation curve which is determined by $T(r, \lambda) = L$, this leads us to investigate the graph of $T(r, \lambda)$.

About the gradient blow-up curve, the following result is well known (define $\frac{B}{C} = 0$ if $C = +\infty$).

Lemma 3.1 ([13]). *Assume condition (1.3) holds. Let $B < +\infty$ and $r(\lambda) = F^{-1}(\frac{B}{\lambda})$. Then the following assertions hold:*

- (a) *The function $r(\lambda)$ is well defined on $(\frac{B}{C}, +\infty)$ and strictly decreasing.*
- (b) *$\lim_{\lambda \rightarrow \frac{B}{C}} r(\lambda) = A$ and $\lim_{\lambda \rightarrow \infty} r(\lambda) = 0$.*

Lemma 3.2 ([13]). *Assume conditions (1.2) and (1.3) hold. Let $T(r, \lambda)$ be defined in (3.1). Then for fixed $r \in I$, $T(r, \lambda)$ is strictly decreasing in λ and $\lim_{\lambda \rightarrow 0} T(r, \lambda) = +\infty$.*

Lemma 3.3 ([13]). *Assume conditions (1.2) and (1.3) hold. Let $T(r, \lambda)$ be defined in (3.1). If $B = +\infty$, then $\lim_{\lambda \rightarrow +\infty} T(r, \lambda) = 0$ for any $r \in (0, A)$.*

The following lemma, which is a generalization of Habets and Omari [6, Lem 3.1], gives smoothness of T with respect to r .

Lemma 3.4 ([13]). *If for any $r \in I$ there exists a locally bounded function $K(r) > 0$ such that for every $s \in (\frac{1}{2}, 1)$,*

$$\left| \frac{f(r) - sf(rs)}{F(r) - F(rs)} \right| \leq K(r), \quad (3.3)$$

then T is differentiable at each point $r \in I$, with derivative

$$\begin{aligned} \frac{\partial T}{\partial r} &= \int_0^1 \frac{1}{\Phi^{-1}(\lambda [F(r) - F(rs)])} ds \\ &\quad - \lambda r \int_0^1 \frac{f(r) - sf(rs)}{[\Phi^{-1}(\lambda [F(r) - F(rs)])]^2 \Phi'(\Phi^{-1}(\lambda [F(r) - F(rs))])} ds \end{aligned} \quad (3.4)$$

and $\frac{\partial T}{\partial r}(r, \lambda)$ is continuous at each point $(r, \lambda) \in I \times (0, \infty)$.

Notice that the interval $(\frac{1}{2}, 1)$ can be replaced with any $(c, 1), c \in (0, 1)$.

3.2. More properties on time maps. For simplicity, in what follows, we usually denote $T(r, \lambda)$ by $T(r)$, $\frac{\partial T}{\partial r}$ by T' , and $\frac{\partial T}{\partial \lambda}$ by T_λ .

Theorem 3.5. Assume that (1.2)–(1.6) hold. Then $T'(r) < 0$ for all $r \in I$.

Proof. Since $f \in C^1$ implies (3.3), T is differentiable. By (3.4), we have

$$T'(r) = \int_0^1 \frac{[\Phi^{-1}(\lambda [F(r) - F(rs)])]^2 \varphi'(\Phi^{-1}(\lambda [F(r) - F(rs)])) - \lambda r [f(r) - sf(rs)]}{[\Phi^{-1}(\lambda [F(r) - F(rs)])]^3 \varphi'(\Phi^{-1}(\lambda [F(r) - F(rs)]))} ds. \quad (3.5)$$

Denote the numerator of the integrand by

$$H(s) = [\Phi^{-1}(\lambda [F(r) - F(rs)])]^2 \varphi'(\Phi^{-1}(\lambda [F(r) - F(rs)])) - \lambda r [f(r) - sf(rs)].$$

Clearly, $H(1) = 0$. Moreover, for $s \in (0, 1)$, we have

$$\begin{aligned} H'(s) &= -2\lambda f(rs)r - \lambda f(rs)r \frac{\Phi^{-1}(\lambda [F(r) - F(rs)])\varphi''(\Phi^{-1}(\lambda [F(r) - F(rs)]))}{\varphi'(\Phi^{-1}(\lambda [F(r) - F(rs)]))} + \lambda f(rs)r + \lambda f'(rs)r^2s \\ &= -\lambda f(rs)r \frac{\Phi^{-1}(\lambda [F(r) - F(rs)])\varphi''(\Phi^{-1}(\lambda [F(r) - F(rs)]))}{\varphi'(\Phi^{-1}(\lambda [F(r) - F(rs)]))} + \lambda r[f'(rs)rs - f(rs)] > 0, \end{aligned}$$

where the last inequality holds due to (1.2)–(1.6). Hence, $H(s) < 0$ on $[0, 1)$. Then $T'(r) < 0$ on I . \square

The following two propositions give some important information of $T(r)$ at the left endpoint of I .

Proposition 3.6. Let $\lambda > 0$ be fixed. Assume conditions (1.2) and (1.3) hold. If $f(0) = 0$ and $0 < \lim_{r \rightarrow 0} \frac{f(r)}{r^\alpha} = E < +\infty$ for some $\alpha > 0$, then

$$\lim_{r \rightarrow 0} T(r) = \begin{cases} 0, & \text{if } 0 < \alpha < 1; \\ \frac{\pi}{2} \sqrt{\frac{\varphi'(0)}{\lambda E}}, & \text{if } \alpha = 1; \\ +\infty, & \text{if } \alpha > 1. \end{cases}$$

Proof. Letting $u = rs$ in (3.1), we obtain

$$T(r) = r \int_0^1 \frac{1}{\Phi^{-1}(\lambda [F(r) - F(rs)])} ds = r^{\frac{1+\alpha}{2}} \int_0^1 \frac{r^{\frac{1+\alpha}{2}}}{\Phi^{-1}(\lambda [F(r) - F(rs)])} ds. \quad (3.6)$$

Since $f(0) = 0$ and $\lim_{r \rightarrow 0} \frac{f(r)}{r^\alpha} = E$, we obtain

$$\frac{F(r) - F(rs)}{r^{\alpha+1}} = \frac{1}{r^{\alpha+1}} \int_{rs}^r f(z) dz = \int_s^1 \frac{f(r\tau)}{(r\tau)^\alpha} \tau^\alpha d\tau \rightarrow \frac{E(1-s^{\alpha+1})}{\alpha+1} \quad \text{as } r \rightarrow 0. \quad (3.7)$$

Moreover, we know

$$\lim_{y \rightarrow 0} \frac{y}{(\Phi^{-1}(y))^2} = \lim_{z \rightarrow 0} \frac{\Phi(z)}{z^2} = \lim_{z \rightarrow 0} \frac{z\varphi'(z)}{2z} = \frac{\varphi'(0)}{2} > 0. \quad (3.8)$$

By the proof of Lemma 2.1 of [13], we have

$$\lim_{r \rightarrow 0} \int_0^1 \frac{r^{\frac{1+\alpha}{2}}}{\Phi^{-1}(\lambda [F(r) - F(rs)])} ds = \int_0^1 \lim_{r \rightarrow 0} \frac{r^{\frac{1+\alpha}{2}}}{\Phi^{-1}(\lambda [F(r) - F(rs)])} ds.$$

This, together with (3.7) and (3.8), implies that

$$\begin{aligned} \lim_{r \rightarrow 0} \int_0^1 \frac{r^{\frac{1+\alpha}{2}}}{\Phi^{-1}(\lambda [F(r) - F(rs)])} ds &= \int_0^1 \lim_{r \rightarrow 0} \frac{\sqrt{\lambda [F(r) - F(rs)]}}{\Phi^{-1}(\lambda [F(r) - F(rs)])} \frac{r^{\frac{1+\alpha}{2}}}{\sqrt{\lambda [F(r) - F(rs)]}} ds \\ &= \int_0^1 \sqrt{\frac{\varphi'(0)(\alpha+1)}{2\lambda E(1-s^{\alpha+1})}} ds. \end{aligned} \quad (3.9)$$

If $\alpha \in (0, 1)$, we have

$$\int_0^1 \sqrt{\frac{\varphi'(0)(\alpha+1)}{2\lambda E(1-s^{\alpha+1})}} ds \leq \int_0^1 \sqrt{\frac{\varphi'(0)(\alpha+1)}{2\lambda E(1-s)}} ds = \sqrt{\frac{2\varphi'(0)(\alpha+1)}{\lambda E}}.$$

Then (3.6) and (3.9) imply $\lim_{r \rightarrow 0} T(r) = 0$.

If $\alpha = 1$, we get

$$\int_0^1 \sqrt{\frac{\varphi'(0)(\alpha+1)}{2\lambda E(1-s^{\alpha+1})}} ds = \int_0^1 \sqrt{\frac{\varphi'(0)}{\lambda E(1-s^2)}} ds = \frac{\pi}{2} \sqrt{\frac{\varphi'(0)}{\lambda E}}.$$

Then (3.6) and (3.9) imply that $\lim_{r \rightarrow 0} T(r) = \frac{\pi}{2} \sqrt{\frac{\varphi'(0)}{\lambda E}}$.

If $\alpha > 1$, we obtain

$$\int_0^1 \sqrt{\frac{\varphi'(0)(\alpha+1)}{2\lambda E(1-s^{\alpha+1})}} ds \geq \int_0^1 \sqrt{\frac{\varphi'(0)(\alpha+1)}{2\lambda E}} ds = \sqrt{\frac{\varphi'(0)(\alpha+1)}{2\lambda E}}.$$

Thus $\lim_{r \rightarrow 0} T(r) = +\infty$ follows from (3.6) and (3.9). □

Proposition 3.7. *Let $\lambda > 0$ be fixed. Assume conditions (1.2), (1.3) and (1.5) hold. Then*

$$\lim_{r \rightarrow 0} T(r) = \begin{cases} \frac{\pi}{2} \sqrt{\frac{\varphi'(0)}{\lambda f'(0)}}, & \text{if } f'(0) > 0; \\ +\infty, & \text{if } f'(0) = 0. \end{cases}$$

Proof. Notice that conditions (1.3) and (1.5) imply $f(0) = 0$.

If $f'(0) > 0$, then $\lim_{r \rightarrow 0} \frac{f(r)}{r} = f'(0) > 0$. By Proposition 3.6, we have

$$\lim_{r \rightarrow 0} T(r) = \frac{\pi}{2} \sqrt{\frac{\varphi'(0)}{\lambda f'(0)}}.$$

We next consider the case $f'(0) = 0$. Since

$$\lim_{r \rightarrow 0} \frac{r^2}{F(r)} = \lim_{r \rightarrow 0} \frac{2r}{f(r)} = \lim_{r \rightarrow 0} \frac{2}{f'(r)} = +\infty,$$

it follows that for any $M > 0$, there exists $\delta_M > 0$ such that

$$\frac{r^2}{F(r)} > M, \quad \text{for } 0 < r < \delta_M.$$

Then for all $s \in [0, 1]$, we have

$$\frac{r^2}{F(r) - F(rs)} \geq \frac{r^2}{F(r)} > M, \quad \text{for } 0 < r < \delta_M. \quad (3.10)$$

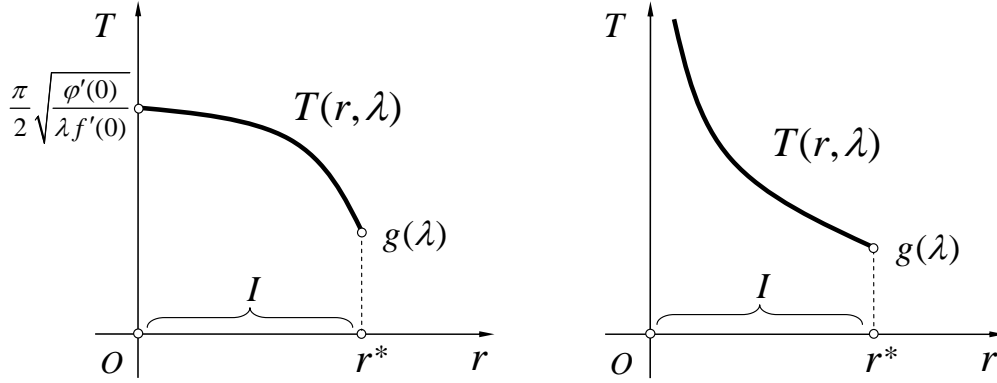


FIGURE 13. The graphs of time map T for $f(0) = 0$ when λ is fixed. Left: $f'(0) > 0$. Right: $f'(0) = 0$.

This, together with (3.6), implies

$$\begin{aligned} T(r) &= \int_0^1 \frac{\sqrt{\lambda[F(r) - F(rs)]}}{\Phi^{-1}(\lambda[F(r) - F(rs)])} \frac{r}{\sqrt{\lambda[F(r) - F(rs)]}} ds \\ &\geq \sqrt{\frac{M}{\lambda}} \int_0^1 \frac{\sqrt{\lambda[F(r) - F(rs)]}}{\Phi^{-1}(\lambda[F(r) - F(rs)])} ds. \end{aligned} \quad (3.11)$$

From (3.8), it follows that

$$\lim_{r \rightarrow 0} \int_0^1 \frac{\sqrt{\lambda[F(r) - F(rs)]}}{\Phi^{-1}(\lambda[F(r) - F(rs)])} ds = \sqrt{\frac{\varphi'(0)}{2}}.$$

This, together with (3.10) and (3.11), implies $\lim_{r \rightarrow 0} T(r) = +\infty$. \square

3.3. The function g . By Theorem 3.5 and Proposition 3.7, we know that the graph of $T(r, \lambda)$ looks like the left or the right in Fig.13. From which, we find that the position of T at the right endpoint r^* of interval I is crucial in the existence of solutions for $T(r, \lambda) = L$. We next discuss how the position of $T(r^*, \lambda)$ changes as the parameter λ varies through positive values. To analyze the behaviors of T at r^* , the same as in [13], we define

$$g(\lambda) = \lim_{r \rightarrow r^*} T(r, \lambda).$$

From (3.2), it follows that

$$g(\lambda) = \begin{cases} \lim_{r \rightarrow A^-} T(r, \lambda), & \text{if } B = +\infty; \\ T(F^{-1}(\frac{B}{\lambda}), \lambda), & \text{if } B < +\infty \text{ and } C = +\infty; \\ \lim_{r \rightarrow A^-} T(r, \lambda), & \text{if } B < +\infty, C < +\infty, \text{ and } \lambda \leq \frac{B}{C}; \\ T(F^{-1}(\frac{B}{\lambda}), \lambda), & \text{if } B < +\infty, C < +\infty, \text{ and } \lambda > \frac{B}{C}. \end{cases} \quad (3.12)$$

The graph of g has been well investigated in [13, Sec 5]. Let us recall some useful results.

Case I: $A, B, C = +\infty$

In this case, from Theorem 4.2 of [13], we know that if condition (2.1) holds, then

$$g(\lambda) = \lim_{r \rightarrow +\infty} T(r, \lambda) \equiv 0.$$

In particular, similar to Corollary 4.4 of [13], we also have that if the range of φ is bounded or $\frac{F(z)}{f(z)}$ is bounded for sufficiently large z , then $g \equiv 0$.

Case II: $A < +\infty, B = +\infty$ and $C = +\infty$

In this case, from Corollary 4.3 of [13], we obtain that $g(\lambda) = \lim_{r \rightarrow A^-} T(r, \lambda) \equiv 0$.

Case III: $A < +\infty$, $B = +\infty$ and $C < +\infty$

In this case, we have

$$g(\lambda) = \lim_{r \rightarrow A^-} T(r, \lambda) = \int_0^C \frac{1}{\Phi^{-1}(\lambda y)} \frac{1}{f \circ F^{-1}(C - y)} dy.$$

By Proposition 5.1 of [13], it is easy to see

Lemma 3.8. *Let $A < +\infty$, $B = +\infty$ and $C < +\infty$. Assume conditions (1.2)–(1.5) hold. Let $g(\lambda)$ be defined in (3.12). Then the following assertions hold:*

- (a) $g(\lambda) > 0$ for all $\lambda > 0$ and g is strictly decreasing in λ .
- (b) $\lim_{\lambda \rightarrow 0} g(\lambda) = +\infty$ and $\lim_{\lambda \rightarrow +\infty} g(\lambda) = 0$.
- (c) For every $L \in (0, +\infty)$ there exists a unique λ_* such that $g(\lambda_*) = L$. Moreover, λ_* is strictly decreasing with respect to L .

Cases IV and V: $B < +\infty$ and $C = +\infty$

In these cases, we have

$$g(\lambda) = T\left(F^{-1}\left(\frac{B}{\lambda}\right)\right) = \int_0^{F^{-1}(\frac{B}{\lambda})} \frac{1}{\Phi^{-1}(B - \lambda F(u))} du \quad (3.13)$$

$$= \int_0^B \frac{1}{\Phi^{-1}(B - y)} \frac{1}{\lambda f(F^{-1}(\frac{y}{\lambda}))} dy. \quad (3.14)$$

Due to the separation of Φ^{-1} and F , it is usually more convenient to use the expression (3.14) to analyze the shape of g .

The next lemma is very useful for computing the limits of g at $+\infty$ and 0.

Lemma 3.9 ([13]). *Let $B < +\infty$ and $C = +\infty$. Assume conditions (1.2), (1.3) and $K := \int_0^B \frac{1}{y \Phi^{-1}(B - y)} dy < +\infty$ hold. Then the following assertions hold:*

- (a) If $\lim_{t \rightarrow 0} \frac{F(t)}{f(t)} = 0$, then $\lim_{\lambda \rightarrow +\infty} g(\lambda) = 0$.
- (b) If $\lim_{t \rightarrow A} \frac{F(t)}{f(t)} = D \in [0, +\infty]$, then $\lim_{\lambda \rightarrow 0} g(\lambda) = DK$.

About some concrete functions φ satisfying $K < +\infty$, see Examples 2.2 and 2.7.

The next result gives a sufficient condition for the monotonicity of g .

Lemma 3.10 ([13]). *Let $B < +\infty$ and $C = +\infty$. Assume conditions (1.2) and (1.3) hold. If f is of class C^1 on $(0, A)$ satisfying*

$$f'(t)F(t) \leq (\lesssim) f^2(t) \quad \text{for } t \in (0, A), \quad (3.15)$$

then $g(\lambda)$ is (strictly) decreasing in λ .

See Example 2.8 for some concrete functions f satisfying (3.15).

When g is not monotone, the situation for g in Case IV may be quite complicated. In Section 2, we have given a brief introduction to g . In particular, we establish a classification of g based on the number of local extreme points in $(0, +\infty)$, the local extreme values, and the limits at 0 and $+\infty$ (see Definition 2.5). Various types of g are illustrated in Fig.2.

Letting $\lambda = \frac{B}{F(r)}$ in (3.13), we transform $g(\lambda)$ to

$$\tilde{g}(r) = g(\lambda)|_{\lambda = \frac{B}{F(r)}} = \int_0^r \frac{1}{\Phi^{-1}(B - B \frac{F(u)}{F(r)})} du. \quad (3.16)$$

Notice that when we plot the graph of T under the coordinate system (r, T) , we actually need the shape of $\tilde{g}(r)$ (not $g(\lambda)$!), because the curve of $\tilde{g}(r)$ (i.e. $g(\frac{B}{F(r)})$) is just the path along which the right endpoint of T moves as λ varies through positive values. See Figs.2 and 14 for a comparison.

Remark 3.1. Consider an important example $\varphi = \varphi_3$, i.e., the mean curvature equation (1.9). Then $B = 1$ and we obtain from (3.14) and (3.16)

$$g(\lambda) = \int_0^1 \frac{y}{\sqrt{1-y^2}} \frac{1}{\lambda f(F^{-1}(\frac{y}{\lambda}))} dy \quad (3.17)$$

and

$$\tilde{g}(r) := g(\lambda)|_{\lambda=\frac{1}{F(r)}} = \int_0^r \frac{F(u)}{\sqrt{F(r)^2 - F(u)^2}} du. \quad (3.18)$$

In Figs.15 and 16, we give some numerical simulations of $\tilde{g}(r)$ in Cases IV and V. For these $\tilde{g}(r)$, the corresponding functions $g(\lambda)$ arise in Examples 2.10–2.22. From Figs.15 and 16, using the monotonicity of $\lambda = \frac{1}{F(r)}$, one can indirectly obtain the numerical results of $g(\lambda)$. As shown in Figs.1, 3–10, the various types of g lead to the rich diversity of bifurcation diagrams for problem (1.1).

Case VI: $A, B, C < +\infty$

In this case, we have

$$g(\lambda) = \begin{cases} \lim_{r \rightarrow A^-} T(r, \lambda) = \int_0^C \frac{1}{\Phi^{-1}(\lambda y)} \frac{1}{f \circ F^{-1}(C-y)} dy, & \text{if } \lambda \leq \frac{B}{C}; \\ T\left(F^{-1}\left(\frac{B}{\lambda}\right), \lambda\right) = \int_0^B \frac{1}{\Phi^{-1}(B-y)} \frac{1}{\lambda f(F^{-1}(\frac{y}{\lambda}))} dy, & \text{if } \lambda > \frac{B}{C}. \end{cases} \quad (3.19)$$

Lemma 3.11 ([13]). Let $A, B, C < +\infty$. Assume conditions (1.2) and (1.3) hold. Then $g(\lambda) > 0$ and g is continuous for all $\lambda > 0$.

By Proposition 5.1 of [13], we know

Lemma 3.12. Let $A, B, C < +\infty$. Assume conditions (1.2)–(1.5) hold. Then g is strictly decreasing on $(0, \frac{B}{C})$ and $\lim_{\lambda \rightarrow 0} g(\lambda) = +\infty$.

Moreover, we know

Lemma 3.13 ([13]). Let $A, B, C < +\infty$. Assume conditions (1.2), (1.3), $\lim_{t \rightarrow 0} \frac{F(t)}{f(t)} = 0$ and $\int_0^B \frac{1}{y \Phi^{-1}(B-y)} dy < +\infty$ hold. Then $\lim_{\lambda \rightarrow +\infty} g(\lambda) = 0$.

See e.g., Type γ_0 of Case VI in Fig.2. Also see the bottom of Fig.16 for numerical examples of $g(\tilde{r})$ in Case VI, which arise in Example 2.23.

4. PROOFS OF MAIN THEOREMS

In this section, we prove the main results which are stated in Section 2.

Note that when $f'(0) > 0$, $\lambda_1 := \frac{\varphi'(0)}{f'(0)} (\frac{\pi}{2L})^2$ is strictly decreasing with respect to L , and further from Proposition 3.7, it follows that $\lim_{r \rightarrow 0} T(r, \lambda_1) = L$.

The proof of Theorem 2.1 Since positive solutions of (1.1) are always symmetric, it implies that $u'(0) = 0$. Thus the existence of solutions of (1.1) is equivalent to that of

$$-\varphi'(u')u'' = \lambda f(u), \quad u'(0) = 0, \quad u(L) = 0.$$

By the definition of the time map, for given $L, \lambda > 0$, the number of positive solutions of (1.1) is precisely the number of solutions of $T(\lambda, r) = L$ for $r \in (0, r^*)$. Here, r^* is the right endpoint of I .

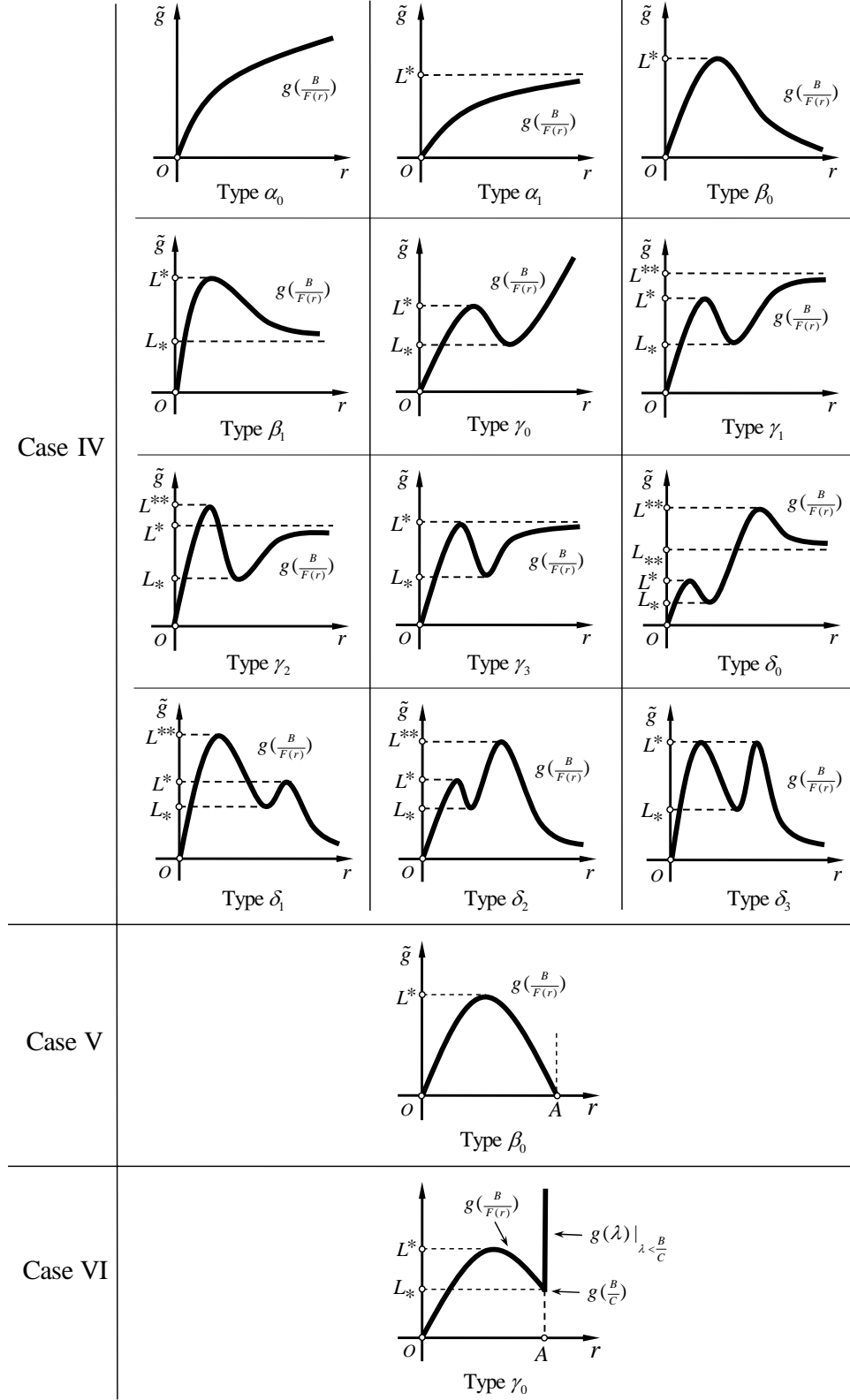


FIGURE 14. Some shapes of $g(\lambda)$ in the coordinate system (r, \tilde{g}) , i.e. the shapes of $g(\frac{B}{F(r)})$. See Fig.2 for a comparison.

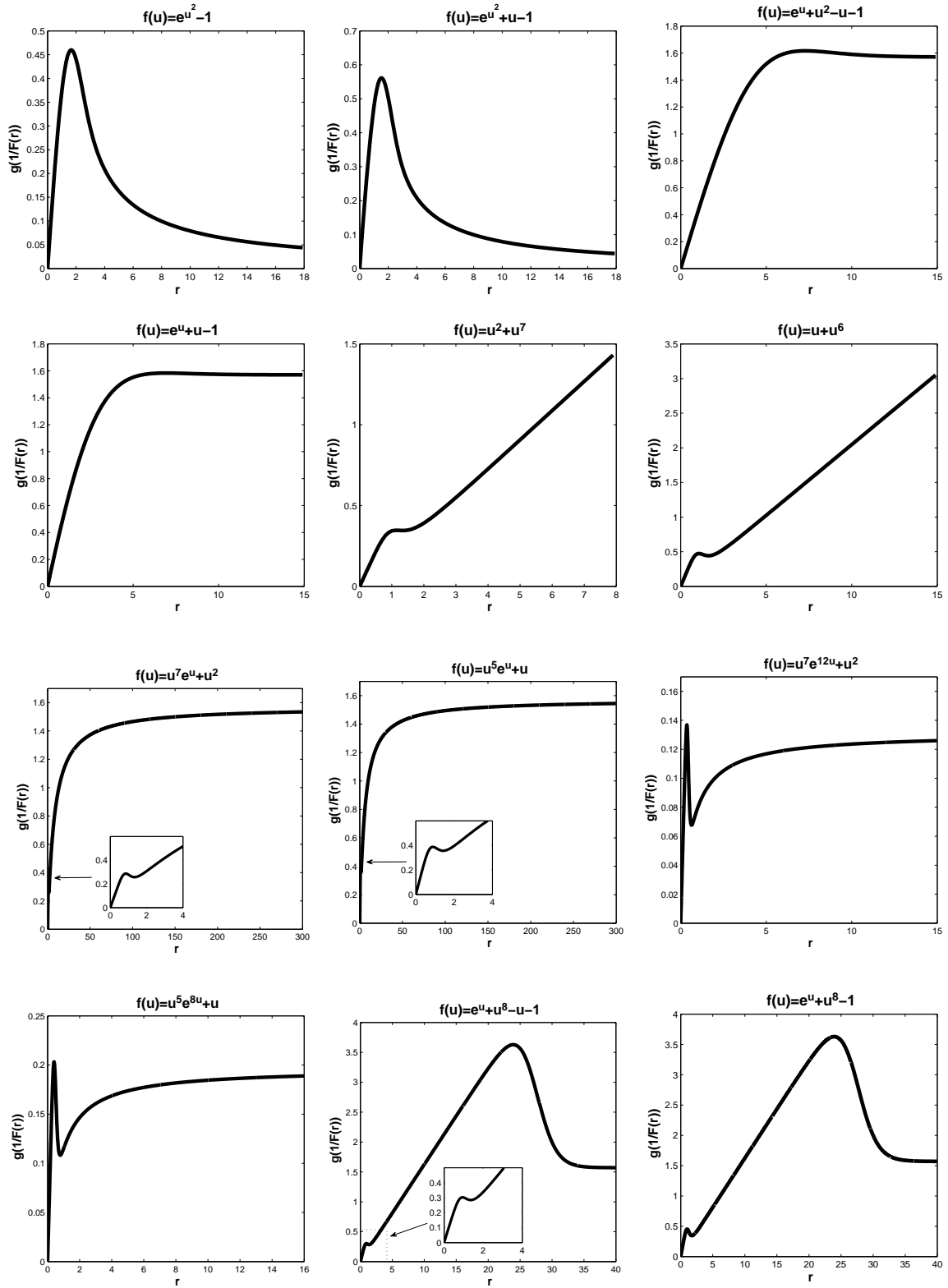


FIGURE 15. Some numerical simulations of $g(\frac{1}{F(r)})$ for (φ_3, f) , i.e., the mean curvature equation, in Examples 2.12–2.18.

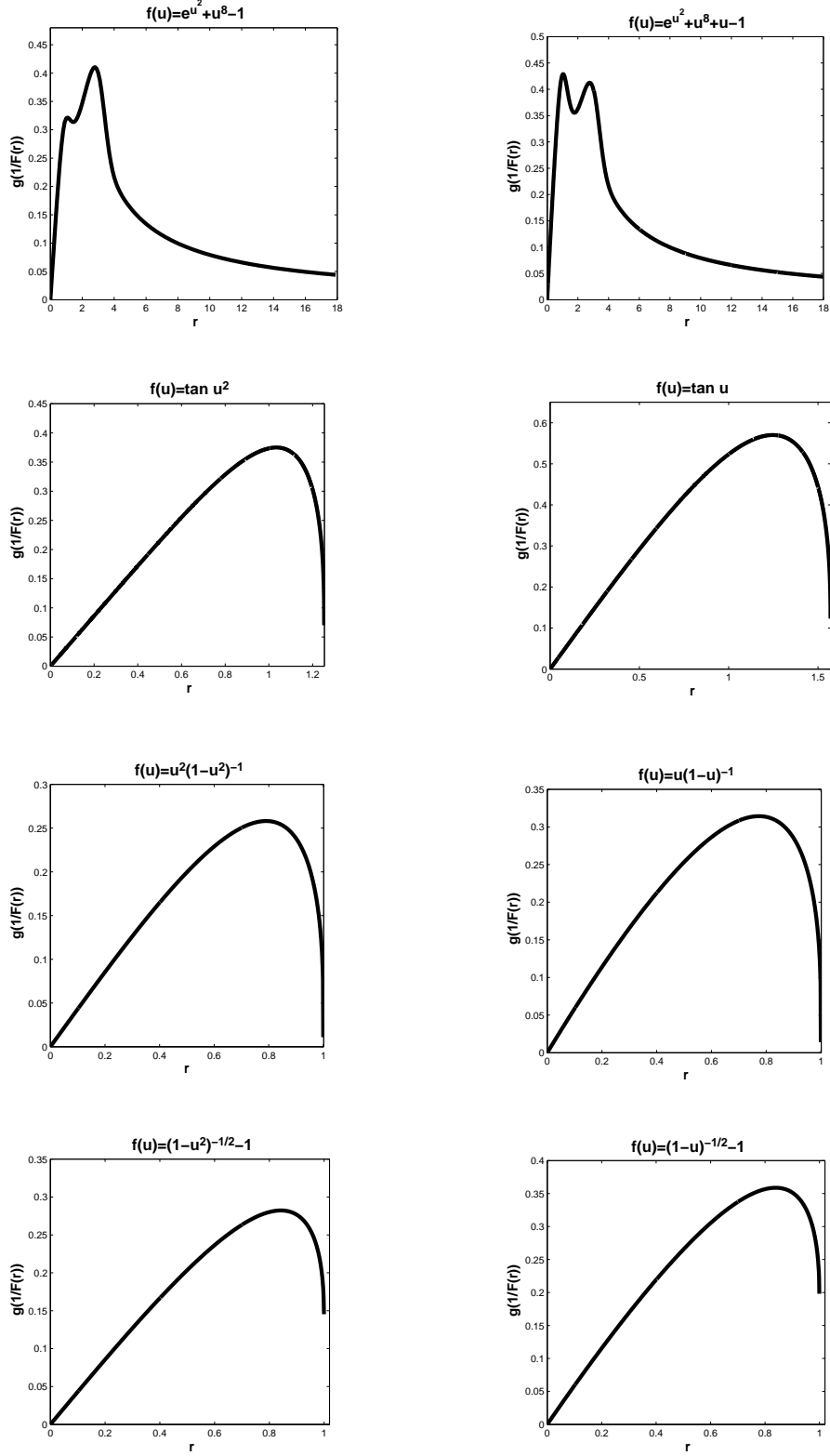


FIGURE 16. Some numerical simulations of $g(\frac{1}{F(r)})$ for (φ_3, f) , i.e., the mean curvature equation, in Examples 2.19, 2.20 and 2.23. Left: $f'(0) = 0$. Right: $f'(0) > 0$.

From Theorem 3.5, we get that $T' < 0$ for all r . Hence for any $\lambda > 0$, $T(\lambda, r) = L$ has at most one solutions in $(0, r^*)$. This completes the proof. \square

The proof of Corollary 2.2 Notice that if f is of class C^2 on $[0, A)$ satisfying conditions (1.3) and $f(0) = 0$, then the convex condition $f''(u) \geq (>)0$ implies the superlinear condition $f'(u)u \geq (>)f(u)$.

Indeed, let $\psi(u) = f'(u)u - f(u)$, then $\psi(0) = 0$ and

$$\psi'(u) = f''(u)u + f'(u) - f'(u) = f''(u)u \geq (>)0.$$

So we can replace (1.5) and (1.6) by (1.5') and (1.6'). Applying Theorem 3.5, we get the result. \square

The proof of Theorem 2.3, Corollary 2.4 and Theorem 2.5 By Theorem 3.5 and Proposition 3.7, we obtain the shape of $T(r, \lambda)$ for fixed $\lambda > 0$. Further, since $g \equiv 0$ (see Section 3.3), by Lemmas 3.2 and 3.3, we obtain the behavior of $T(r, \lambda)$ when λ varies through positive values (see Fig.17).

In particular, for given $L > 0$, we have

- (1) If $f'(0) = 0$, then for any $\lambda > 0$ there exists a unique $r \in (0, r^*)$ such that $T(r, \lambda) = L$,
- (2) If $f'(0) > 0$, then $\lim_{r \rightarrow 0} T(r, \lambda_1) = L$ and for any $\lambda \in (0, \lambda_1)$ there exists a unique $r \in (0, r^*)$ such that $T(r, \lambda) = L$.

Thus we completes the proof. \square

The proof of Theorem 2.6 The proof is similar to the previous one, a key difference is that g is strictly decreasing (Lemma 3.8). This leads to that for given $L > 0$, there exists a unique $\lambda_* > 0$ such that $g(\lambda_*) = L$. The monotonicity of g implies λ_* is strictly decreasing with respect to L (see Fig.17).

In particular, for given $L > 0$, we have

- (1) There exists a unique $\lambda_* > 0$ such that $T(r, \lambda_*) = L$.
- (2) If $f'(0) = 0$, then for any $\lambda > \lambda_*$ there exists a unique $r \in (0, r^*)$ such that $T(r, \lambda) = L$, while if $f'(0) > 0$, then $\lim_{r \rightarrow 0} T(r, \lambda_1) = L$ and for any $\lambda \in (\lambda_*, \lambda_1)$ there exists a unique $r \in (0, r^*)$ such that $T(r, \lambda) = L$.

Thus we completes the proof. \square

The proof of Theorem 2.7 Since (φ, f) is of Type IV- α_0 , it follows that g is strictly decreasing, $\lim_{\lambda \rightarrow 0} g(\lambda) = +\infty$ and $\lim_{\lambda \rightarrow +\infty} g(\lambda) = 0$ (see Figs.2 and 14). Therefore the proof is the same as that of Theorem 2.6, we omit it (see Fig.18). \square

The proof of Corollary 2.8 From Lemmas 3.9 and 3.10, it follows that (φ, f) is of Type IV- α_0 and hence the conclusions of Theorem 2.7 hold. \square

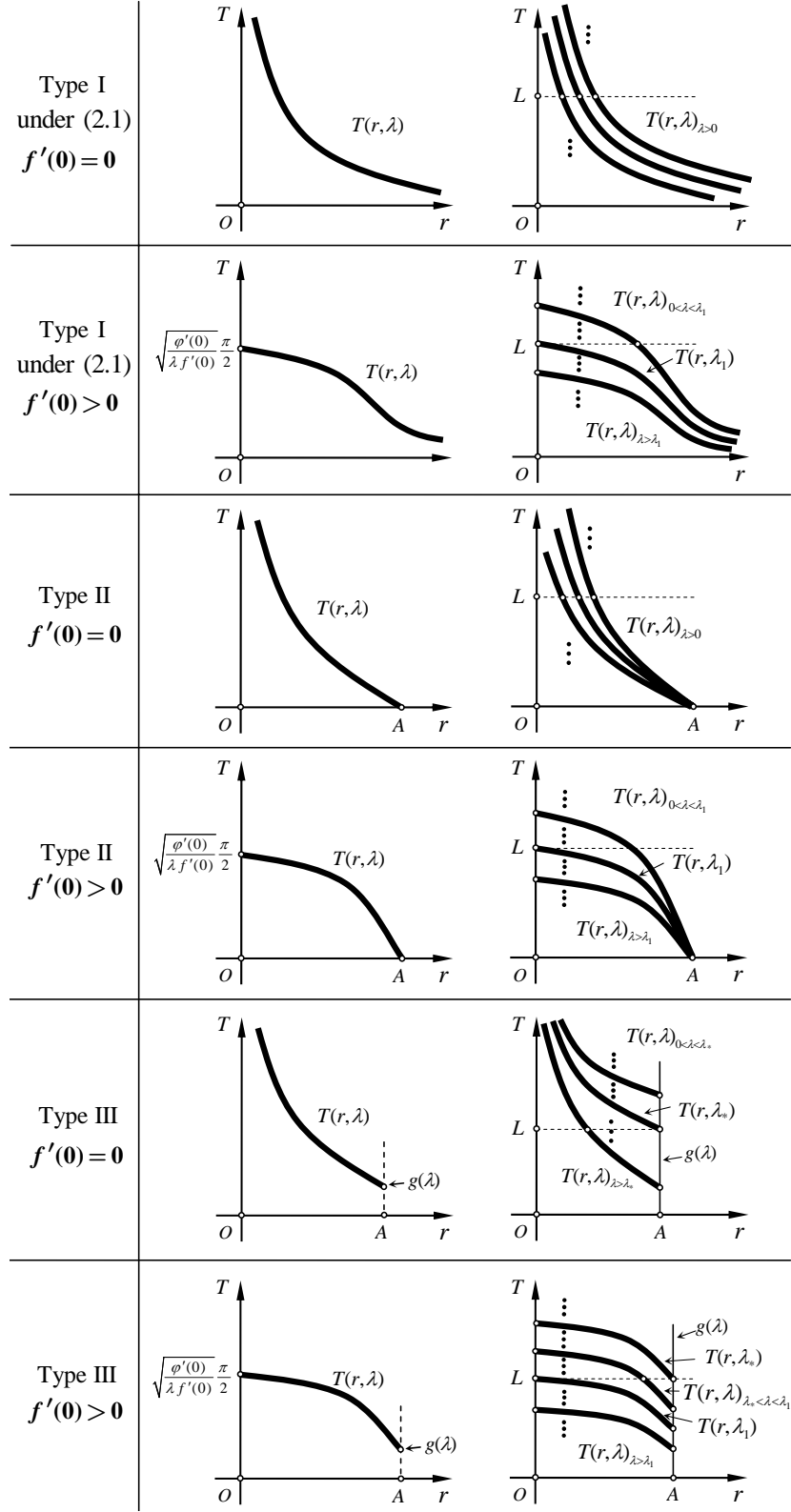
The proof of Theorems 2.9 and 2.10 By Theorem 3.5 and Proposition 3.7, we obtain the shape of $T(r, \lambda)$ for fixed $\lambda > 0$ (see Fig.13 or the left of Type IV- α_0 in Fig.18).

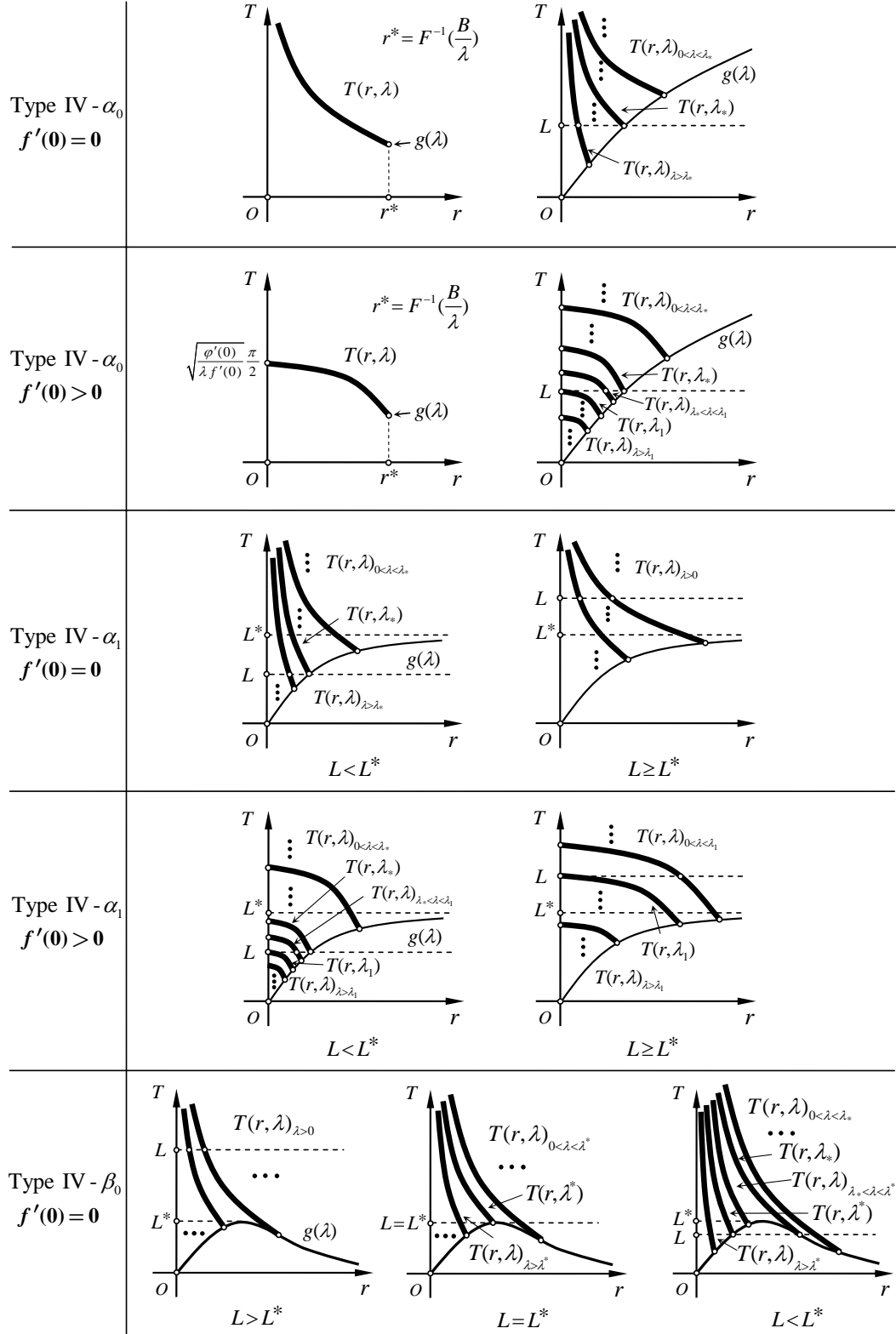
Since (φ, f) is of Type IV- α_1 , it follows that g is strictly decreasing, $\lim_{\lambda \rightarrow 0} g(\lambda) \in (0, +\infty)$ and $\lim_{\lambda \rightarrow +\infty} g(\lambda) = 0$ (see Figs.2 and 14). This, together with Lemma 3.2, gives the behavior of $T(r, \lambda)$ when λ varies through positive values (see the right of Fig.18). In particular, letting $L^* = \sup g$, we have

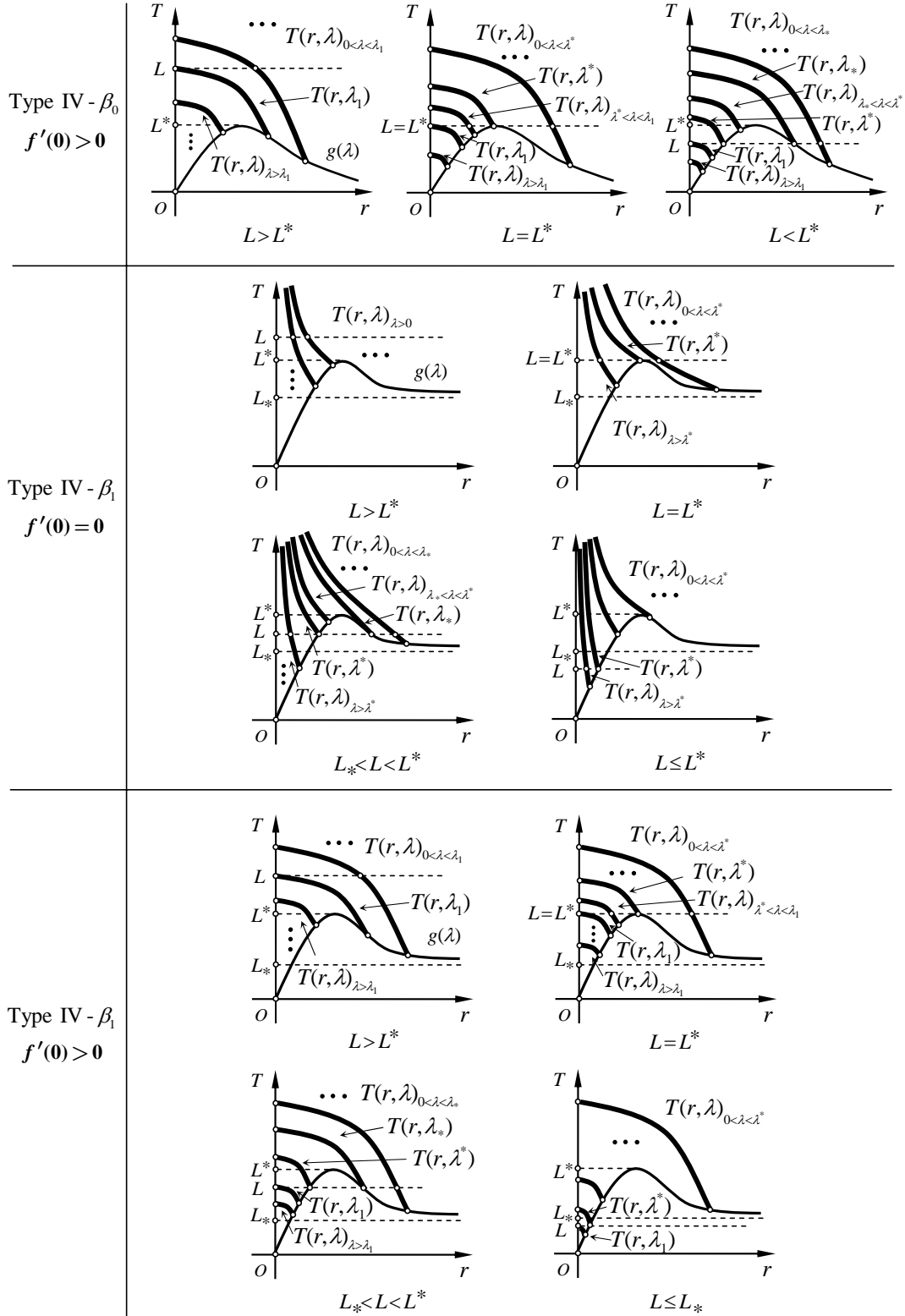
- (1) For given $L \in (0, L^*)$, there exists a unique $\lambda_* > 0$ such that $g(\lambda_*) = L$. Moreover, the monotonicity of g implies that of λ_* with respect to L .
- (2) For given $L \in [L^*, +\infty)$, if $f'(0) = 0$, then for any $\lambda > 0$ there exists a unique $r \in (0, r^*)$ such that $T(r, \lambda) = L$, while if $f'(0) > 0$, then $\lim_{r \rightarrow 0} T(r, \lambda_1) = L$ and for any $\lambda \in (0, \lambda_1)$ there exists a unique $r \in (0, r^*)$ such that $T(r, \lambda) = L$.

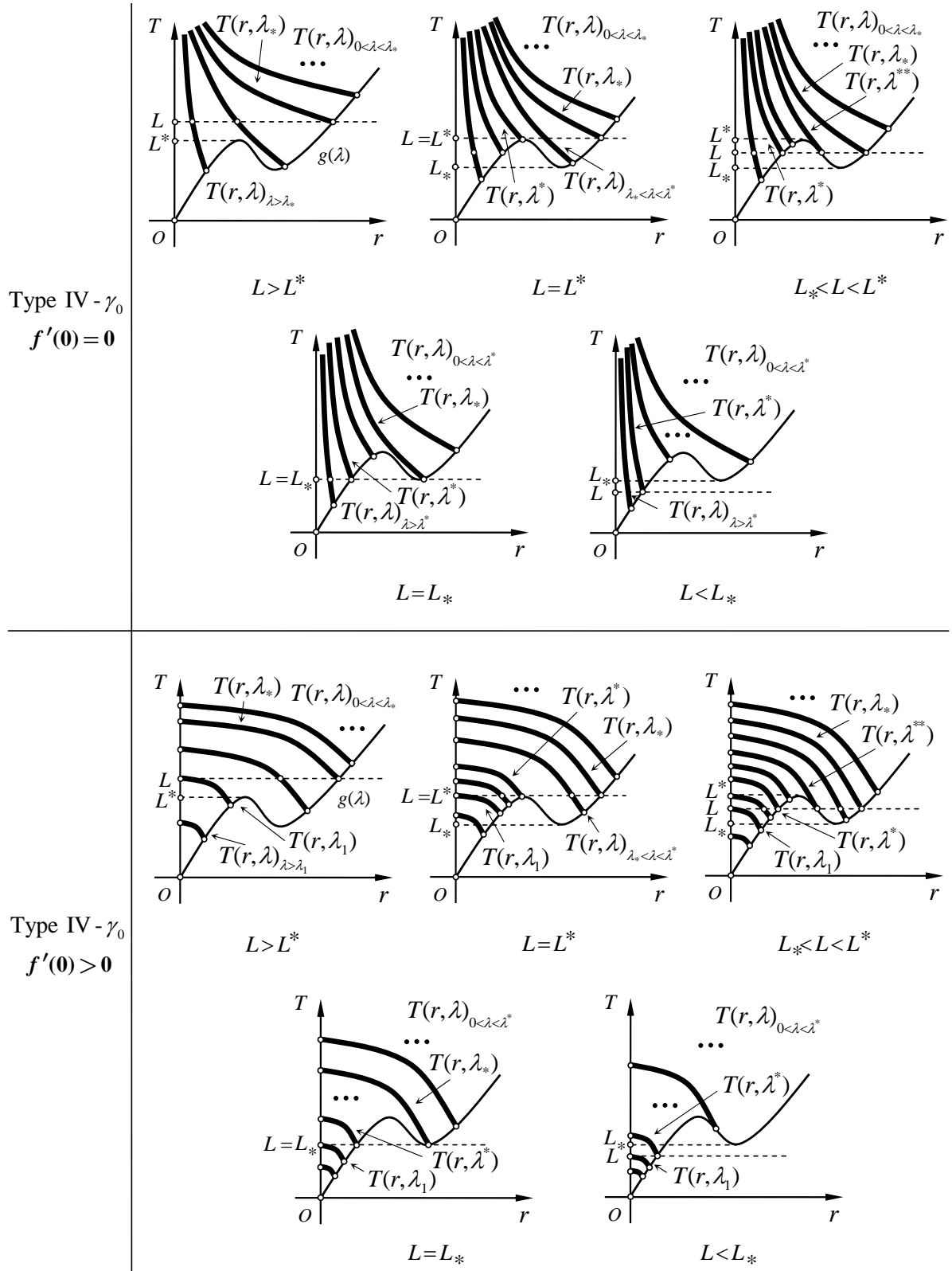
Thus we completes the proof. \square

The proof of Corollary 2.11 From Lemmas 3.9 and 3.10, it follows that (φ, f) is of Type IV- α_1 and hence the conclusions of Theorems 2.9 and 2.10 hold. \square

FIGURE 17. Time maps for Types I-III with $f(0) = 0$ when λ varies.

FIGURE 18. Time maps for Types IV- α_0 , IV- α_1 and IV- β_0 with $f(0) = 0$ when λ varies.

FIGURE 19. Time maps for Types IV- β_0 and IV- β_1 with $f(0) = 0$ when λ varies.

FIGURE 20. Time maps for Type IV- γ_0 with $f(0) = 0$ when λ varies.

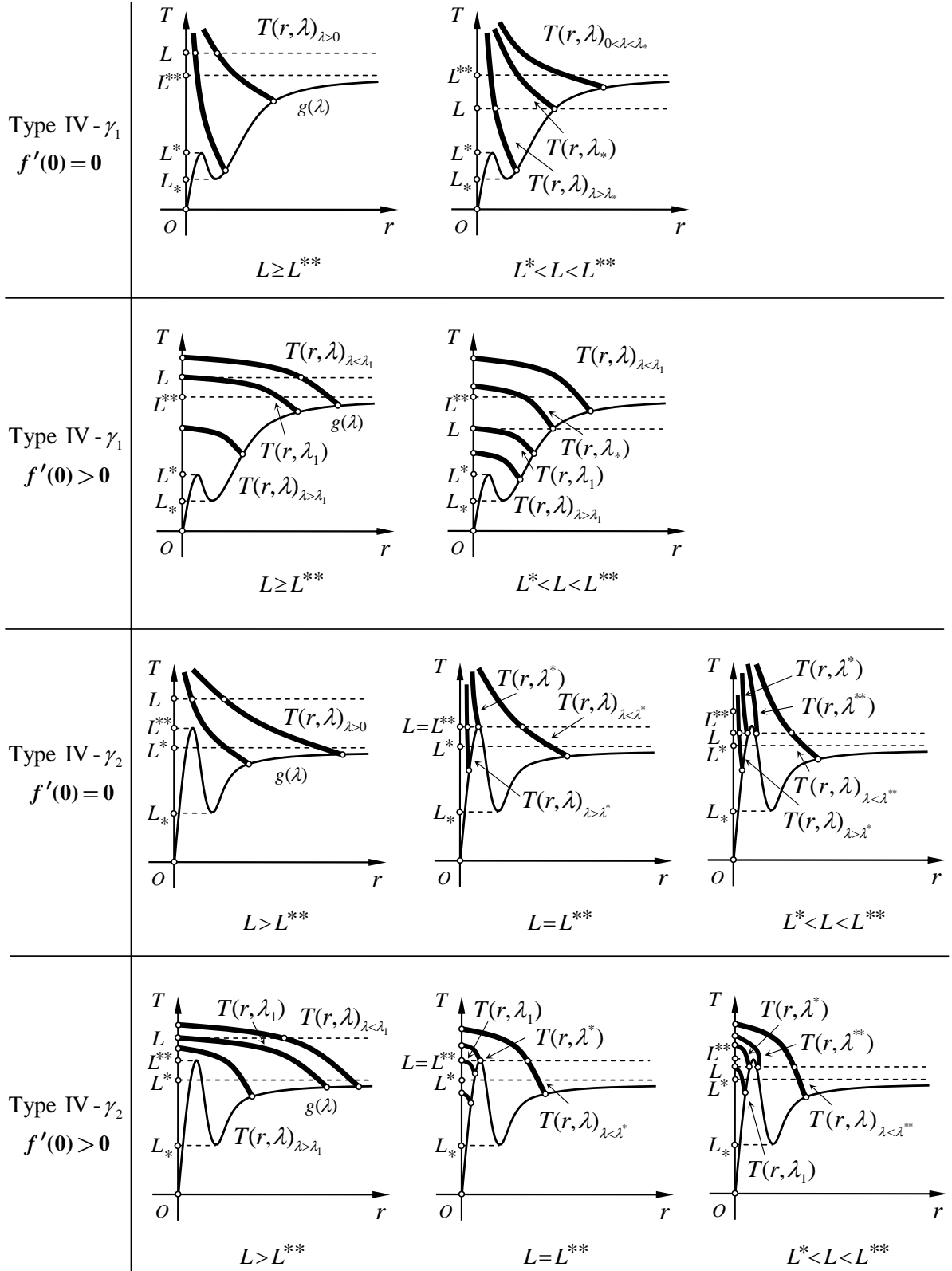


FIGURE 21. Time maps for Types IV- γ_1 and IV- γ_2 with $f(0) = 0$ when λ varies. The remaining cases $L = L^*$, $L_* < L < L^*$, $L = L_*$ and $L < L^*$ are the same as Types IV- γ_0 .

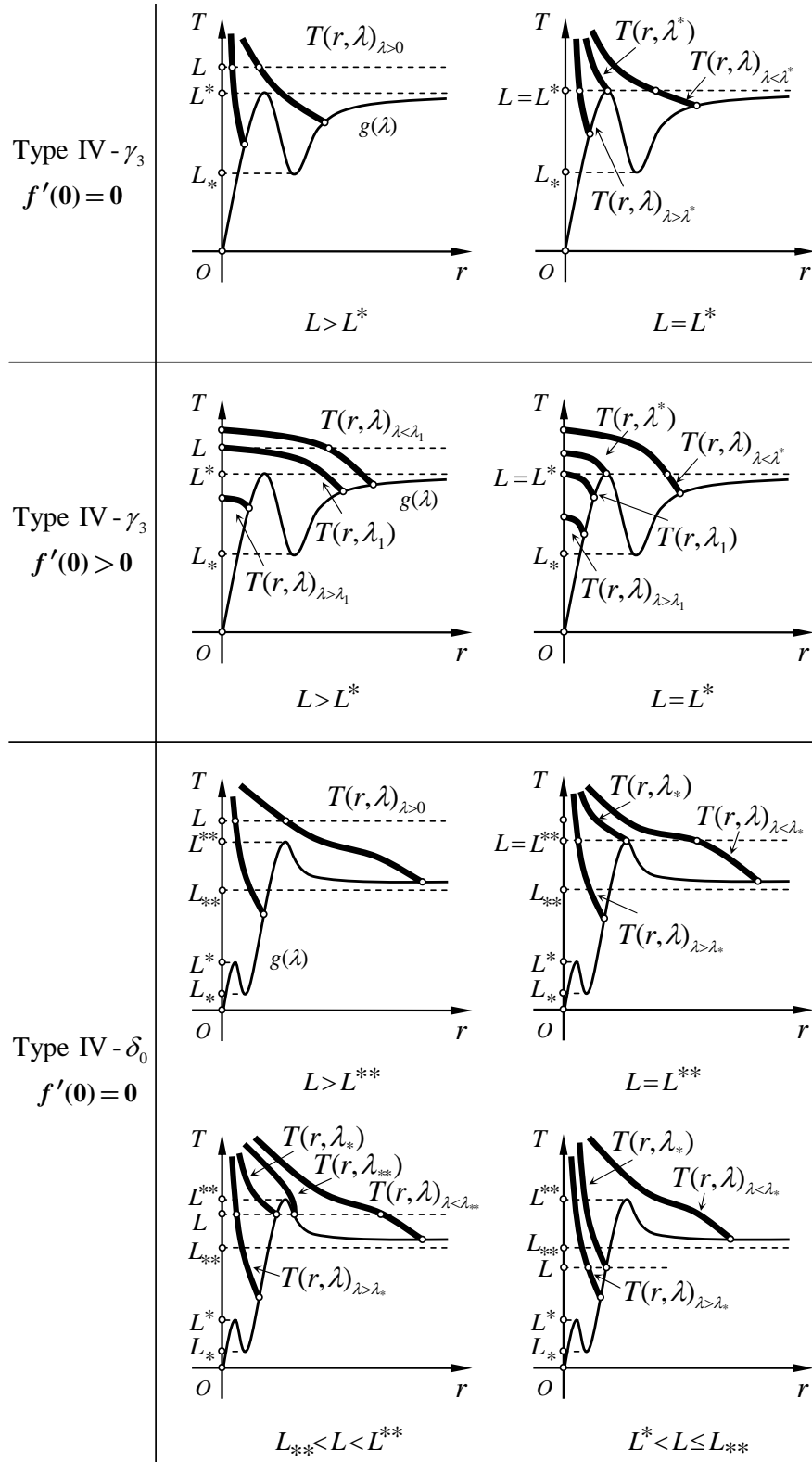


FIGURE 22. Time maps for Types IV- γ_3 and IV- δ_0 with $f(0) = 0$ when λ varies. The remaining cases $L = L^*$, $L_* < L < L^*$, $L = L_*$ and $L < L^*$ are the same as Types IV- γ_0 .

The proof of Theorems 2.12 and 2.13 By Theorem 3.5 and Proposition 3.7, we obtain the shape of $T(r, \lambda)$ for fixed $\lambda > 0$ (see Fig.13 or the left of Type IV- α_0 in Fig.18).

Since (φ, f) is of Type IV- β_0 , we know the graph of g (see Figs.2 and 14). This, together with Lemma 3.2, gives the behavior of $T(r, \lambda)$ when λ varies through positive values (see Figs.18 and 19). In particular, letting $L^* = \sup g$, we have

- (1) For given $L = L^*$, there exists a unique $\lambda_* > 0$ such that $g(\lambda_*) = L$.
- (2) For given $L \in (0, L^*)$, there exist $\lambda^* > \lambda_* > 0$ such that $g(\lambda^*) = g(\lambda_*) = L^*$. Moreover, the shape of g implies the monotonous relations of λ_* and λ^* with respect to L .
- (3) For given $L > 0$, if $f'(0) > 0$, then $\lim_{r \rightarrow 0} T(r, \lambda_1) = L$.
- (4) For $L > L^*$, the situation is similar to Type IV- α_1 .

Thus we complete the proof. \square

The proofs of Theorems 2.14–2.29 All proofs are similar to that of Theorems 2.12 and 2.13, we omit them. Notice that for fixed $\lambda > 0$, the graphs of all $T(r, \lambda)$ in Case IV are similar to the left of Type IV- α_0 in Fig.18, various different types of g (see Fig.14) essentially lead to the differences of bifurcation diagrams.

In Figs.19 and 20, we give the graphs of Time maps for Types IV- β_1 and IV- γ_0 when λ varies. In Figs.21 and 22, we do not give all graphs of Time maps for Types IV- γ_1 , IV- γ_2 , IV- γ_3 and IV- δ_0 because the remaining cases $L = L^*$, $L_* < L < L^*$, $L = L_*$ and $L < L_*$ are the same as Types IV- γ_0 . Types IV- δ_1 , IV- δ_2 and IV- δ_3 can be discussed in the same way, we omit them.

Besides, the discussions for Types V- β_0 and VI- γ_0 are also completely similar to those of Types IV- β_0 and IV- γ_0 (see Figs.13 and 14), we omit them. \square

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REFERENCES

- [1] C. Bereanu and J. Mawhin. Boundary-value problems with non-surjective ϕ -Laplacian and one-sided bounded nonlinearity. *Adv. Differential Equations*, 11(1):35–60, 2006.
- [2] D. Bonheure, P. Habets, F. Obersnel, and P. Omari. Classical and non-classical positive solutions of a prescribed curvature equation with singularities. *Rend. Istit. Mat. Univ. Trieste*, 39:63–85, 2007.
- [3] D. Bonheure, P. Habets, F. Obersnel, and P. Omari. Classical and non-classical solutions of a prescribed curvature equation. *J. Differential Equations*, 243(2):208–237, 2007.
- [4] N. D. Brubaker and J. A. Pelesko. Analysis of a one-dimensional prescribed mean curvature equation with singular nonlinearity. *Nonlinear Anal.*, 75(13):5086–5102, 2012.
- [5] M. Burns and M. Grinfeld. Steady state solutions of a bi-stable quasi-linear equation with saturating flux. *Eur. J. Appl. Math.*, 22(5):317–331, 2011.
- [6] P. Habets and P. Omari. Multiple positive solutions of a one-dimensional prescribed mean curvature problem. *Commun. Contemp. Math.*, 9(5):701–730, 2007.
- [7] K.-C. Hung, Y.-H. Cheng, S.-H. Wang, and C.-H. Chuang. Exact multiplicity and bifurcation diagrams of positive solutions of a one-dimensional multiparameter prescribed mean curvature problem. *J. Differential Equations*, 2014. In press.
- [8] P. Korman and Y. Li. Global solution curves for a class of quasilinear boundary-value problems. *Proc. Roy. Soc. Edinburgh Sect. A*, 140(6):1197–1215, 2010.
- [9] T. Laetsch. The number of solutions of a nonlinear two point boundary value problem. *Indiana Univ. Math. J.*, 20:1–13, 1970.
- [10] H. A. Levine. Quenching, nonquenching, and beyond quenching for solution of some parabolic equations. *Ann. Mat. Pura Appl. (4)*, 155:243–260, 1989.
- [11] W. Li and Z. Liu. Exact number of solutions of a prescribed mean curvature equation. *J. Math. Anal. Appl.*, 367(2):486–498, 2010.
- [12] F. Obersnel. Classical and non-classical sign-changing solutions of a one-dimensional autonomous prescribed curvature equation. *Adv. Nonlinear Stud.*, 7(4):671–682, 2007.

- [13] H. Pan and R. Xing. On the existence of positive solutions for some nonlinear boundary value problems and applications to MEMS models. *Discrete Contin. Dyn. Syst.* (To appear).
- [14] H. Pan and R. Xing. Time maps and exact multiplicity results for one-dimensional prescribed mean curvature equations. *Nonlinear Anal.*, 74(4):1234–1260, 2011.
- [15] H. Pan and R. Xing. Time maps and exact multiplicity results for one-dimensional prescribed mean curvature equations. II. *Nonlinear Anal.*, 74(11):3751–3768, 2011.
- [16] H. Pan and R. Xing. Exact multiplicity results for a one-dimensional prescribed mean curvature problem related to MEMS models. *Nonlinear Anal. Real World Appl.*, 13(5):2432–2445, 2012.
- [17] H. Pan and R. Xing. Applications of total positivity theory to 1D prescribed curvature equations. 2014. Preprint.
- [18] J. A. Pelesko and T. A. Driscoll. The effect of the small-aspect-ratio approximation on canonical electrostatic MEMS models. *J. Engrg. Math.*, 53(3-4):239–252, 2005.
- [19] X. Zhang and M. Feng. Exact number of solutions of a one-dimensional prescribed mean curvature equation with concave-convex nonlinearities. *J. Math. Anal. Appl.*, 395:393–402, 2013.

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